# Geodesic X-Ray Transform on Asymptotically Hyperbolic Manifolds

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**Abstract**

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This dissertation contains work of the author and joint work with C. Robin Graham concerning the geodesic X-ray transform in the setting of asymptotically hyperbolic manifolds. It is divided into three self contained chapters, each addressing a different question. The topic of the first chapter is the local injectivity of the X-ray transform, extending a result proved by Uhlmann and Vasy ([UV16]) on compact manifolds with boundary. Assuming knowledge of the X-ray transform for geodesics contained in a small neighborhood of a boundary point we show local injectivity for asymptotically hyperbolic metrics even modulo $O(\rho^5)$ in dimension 3 and higher. In the second chapter we construct examples of asymptotically hyperbolic metrics demonstrating that in the asymptotically hyperbolic setting absence of conjugate points does not suffice to exclude boundary conjugate points. The construction uses techniques developed by Gulliver ([Gul75]) and clarifies the definition of a simple asymptotically hyperbolic manifold, formulated by Graham, Guillarmou, Stefanov and Uhlmann ([GGS$^+$]). In the third chapter we show a stability estimate for the X-ray transform on simple asymptotically hyperbolic manifolds, extending to this setting the work of Stefanov and Uhlmann on simple compact manifolds with boundary ([SU04]).

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# DEDICATION

To my parents, Thodoris and Tina.

# INTRODUCTION AND STATEMENT OF THE RESULTS

This thesis consists of three self contained chapters addressing questions related to the geodesic X-ray transform, given by

$$If(\gamma) = \int_\gamma f ds, \tag{0.0.1}$$

where $\gamma$ is a geodesic of a Riemannian metric $g$ on a Riemannian manifold and $ds$ denotes integration with respect to $g$-arc length, in the setting of asymptotically hyperbolic manifolds. Typically one has access to the X-ray transform of an unknown function and the goal is to infer as much information as possible regarding $f$: one would wish to know, for example, whether the transform is injective and stable, what is its range, and whether it is possible to have an inversion formula for recovering $f$. As is reasonable to expect, the geometry of the Riemannian manifold greatly influences the answers to such questions, and so do a priori assumptions on $f$, such as ones regarding its regularity and growth. In the case of the 2-dimensional Euclidean space the geodesic X-ray transform is also known as the Radon transform. It lies in the foundation of Computed Tomography and it has been studied extensively since the early 20th century, starting with Radon's 1917 paper ([Rad17]); several classical results on the Radon transform from the point of view of tomography can be found for instance in [Nat86]. The geodesic X-ray transform in various settings has numerous applications, including medical, geophysical and ultrasound imaging. It is especially important on compact manifolds with boundary: among other things, in this setting it is the linearization of the long standing *boundary rigidity problem* over a conformal class of metrics. The boundary rigidity problem is the question of whether a Riemannian metric $g$ on a compact manifold with boundary can be determined by the distance function between boundary points, up to a diffeomorphism fixing the boundary. Partly due to its relevance to the boundary rigidity problem, there is a well developed theory for the X-ray transform on compact manifolds with boundary.

Throughout this dissertation the geometric setting will be a class of non-compact Riemannian manifolds called *asymptotically hyperbolic* (AH). Let $\overline{M}$ be a compact manifold with boundary of dimension $n+1$ and $M$ be its interior. A $C^\infty$ metric $g$ on $M$ is called asymptotically hyperbolic if for some (and hence any) smooth boundary defining function $\rho$ (that is, $\rho\big|_{\partial M} = 0$, $\rho > 0$ on $M$, $d\rho\big|_{\partial M} \not\equiv 0$) the Riemannian metric $\overline{g} := \rho^2 g$ on $M$ extends to a smooth[1] metric on $\overline{M}$, with the additional property that $|d\rho|^2_{\overline{g}} \equiv 1$ on $\partial M$[2]. The classical example of an AH manifold is the Poincaré ball model of the hyperbolic space of constant sectional curvature $-1$, the manifold being the Euclidean unit ball $\mathbb{B}^{n+1} = \{x = (x^0, \ldots, x^n) \in \mathbb{R}^{n+1} : |x| < 1\}$ with the metric

$$h := 4\frac{\sum_{j=0}^n (dx^j)^2}{(1-|x|^2)^2}.$$

Unlike hyperbolic space, AH manifolds need not be symmetric or homogeneous, so the tools used to study them are primarily analytic. Interest in the study of AH manifolds has risen in the past two decades, since the AdS/CFT conjecture, proposed in [Mal98], related conformal field theories with gravity theories on AH spaces.

We proceed to list a few important properties of AH manifolds. Since for any smooth boundary defining function $\rho$ and any $u \in C^\infty(\overline{M})$ the function $\rho e^u$ is still a smooth boundary defining function, $g$ determines a conformal family of metrics on the boundary given by $[\overline{g}\big|_{T\partial M}]$. This conformal class of metrics determined by $g$ is called the *conformal infinity* of $g$. As shown in [Maz86], $(M, g)$ is a geodesically complete Riemannian manifold with sectional curvatures approaching $-(|d\rho|^2_{\overline{g}})\big|_{\partial M} = -1$ as $\rho \to 0$, and this justifies the name asymptotically hyperbolic. Moreover, any geodesic $\gamma(t)$ in an AH manifold that eventually exits every compact set approaches a boundary point as $t \to \infty$ and $\rho \circ \gamma(t) = O(e^{-t})$. In [GL91], Graham and Lee show that for each conformal representative in the conformal infinity of $g$ there exists a unique boundary defining function $\rho$ inducing a product decomposition

[1]Many authors assume less regularity than $C^\infty$ for $\overline{g}$.

[2]Note that $(|d\rho|^2_{\overline{g}})\big|_{\partial M}$ is independent of the choice of boundary defining function.

$[0,\varepsilon) \times \partial M$ of a collar neighborhood of the boundary such that the metric can be written in the form

$$g = \frac{d\rho^2 + h_\rho}{\rho^2}, \tag{0.0.2}$$

where $h_\rho$ is a 1-parameter family of smooth metrics on $\partial M$, smooth in $\rho$ up to $\rho = 0$, $h_0$ being the conformal representative. We say that an AH metric is in *normal form* if it is written as in (0.0.2). Note that for $\rho$ as in (0.0.2) one has $|d\rho|_g^2 = 1$ in a neighborhood of $\partial M$.

Due to some of the properties discussed above, studying the geodesic X-ray transform on an AH manifold poses some interesting challenges. For instance, completeness implies that the integral in (0.0.1) becomes $If(\gamma) = \int_{-\infty}^{\infty} f(\gamma(t))dt$ for any unit speed geodesic $\gamma(t)$ and it might not converge unless some conditions are imposed on the function $f$; it suffices to assume, for instance, that $f \in |\log\rho|^\alpha C^0(\overline{M})$, $\alpha < -1$, provided that the geodesic $\gamma$ eventually exits every compact set as both $t \to \pm\infty$ (i.e. is not *trapped* in either the forward or backward direction). Another issue is related to the parametrization of the space of geodesics, which is of central importance since $If$ is a function on that space. On non-trapping compact manifolds with boundary (that means by definition that all geodesics intersect the boundary twice in finite time), one can parametrize geodesics by their incoming velocities. In the AH setting it is not obvious how something similar can be done, since all geodesics approach the boundary orthogonally (a parametrization of the space geodesics on AH manifolds in a way analogous to the compact manifold with boundary case was defined in [GGS$^+$]; this point will be discussed in more detail in Chapter 3, Section 3.1).

In Chapter 1 we address the question of local injectivity for the X-ray transform on an asymptotically hyperbolic manifold. This means the X-ray transform of a function is known for geodesics staying within a neighborhood of a boundary point and one asks whether it is possible to recover the function there. The positive answer to the corresponding question on compact manifolds with boundary in dimensions 3 and higher by Uhlmann and Vasy ([UV16]) was one of the major breakthroughs of the past decade in the study of the geodesic

X-ray transform. Given a compact manifold with strictly convex boundary of dimension at least 3, they showed that the local geodesic X-ray transform is injective on functions lying in weighted Sobolev spaces and supported in a neighborhood of a boundary point. Moreover, they showed a stability estimate and global injectivity with reconstruction for the X-ray transform, assuming in addition that the manifold can be foliated by strictly convex hypersurfaces.

We describe the main result of Chapter 1, which is the result of a joint work with Robin Graham. As already mentioned, we will focus on (0.0.1) restricted to a subset of geodesics. If $(M, g)$ is AH and $U \subset \overline{M}$ (typically an open neighborhood of a point $p \in \partial\overline{M}$ or its closure), a geodesic is said to be $U$-local if $\gamma(t) \in U$ for all $t \in \mathbb{R}$ and $\lim_{t\to\pm\infty} \gamma(t) \in U \cap \partial\overline{M}$. The set $\Omega_U$ of $U$-local geodesics is nonempty if $U$ is any open neighborhood of a boundary point; this is a consequence of the existence of "short" geodesics (see section 2.2 of [GGS+]). As we will indicate in Section 1.2, for $U$ a small neighborhood of a boundary point, the map $f \to If|_{\Omega_U}$ can be defined on $\rho^{3/2}L^2(U; dv_{\overline{g}})$ with values in an appropriate $L^2$ space (here $dv_{\overline{g}}$ denotes the volume form with respect to the metric $\overline{g}$ on $\overline{M}$).

To state the result we will need a hypothesis on the metric $g$. We say that an AH metric $g$ is *even mod* $O(\rho^N)$, where $N$ is a positive odd integer, if whenever $g$ is written in normal form (0.0.2) in a neighborhood of $\partial\overline{M}$, one has

$$(\partial_\rho)^m h_\rho\big|_{\rho=0} = 0 \text{ for } m \text{ odd, } 1 \leq m < N. \tag{0.0.3}$$

In the case when (0.0.3) holds for any odd $N > 0$ the metric $g$ will be called *even*. As shown in [Gui05, Lemma 2.1], evenness mod $O(\rho^N)$ is a well defined property of an AH metric, independent of the chosen conformal representative determining the normal form (0.0.2).

Our local injectivity result is the following:

**Theorem 1.** *Let $\overline{M}$ be a manifold with boundary of dimension at least 3, with its interior endowed with an asymptotically hyperbolic metric $g$ that is even mod $O(\rho^5)$. Given any neighborhood $V$ in $\overline{M}$ of $p \in \partial\overline{M}$, there exists a neighborhood $U \subset V$ in $\overline{M}$ of $p$ such that $f \to If\big|_{\Omega_U}$ is injective on $\rho^{3/2}L^2(U; dv_{\overline{g}})$.*

Injectivity of the X-ray transform has been studied extensively on various classes of Riemannian manifolds. On compact manifolds with boundary the theory is very well developed. We mention a few important works and refer the reader to [IM] for a thorough survey: classical results can be found in [Muk75], [Muk77], [MR78] and [Sha94]; more recently, important works include [PSU13] on surfaces, [Gui17], and [UV16] on compact manifolds with boundary of dimension at least 3, as already mentioned. For classes of manifolds that overlap with AH ones, there is well developed theory for the X-ray transform on hyperbolic space from the point of view of symmetric spaces ([Hel11]), also see [BC91] and [Bal05]. More recently, the X-ray transform has been studied on Cartan-Hadamard manifolds in [Leh] and [LRS18]. Those are by definition complete, simply connected manifolds of non-positive curvature; they are diffeomorphic to $\mathbb{R}^n$. The first work showing injectivity results for the X-ray transform specifically in the setting of AH manifolds can be found in [GGS$^+$].

Our approach for proving Theorem 1 is motivated by the following observation. Recall that the Klein model for hyperbolic space is another metric on $\mathbb{B}^{n+1}$, obtained from the Poincaré metric by a change of the radial variable. Geodesics for the Klein model are straight line segments in $\mathbb{R}^{n+1}$ under suitable parametrizations. So the hyperbolic X-ray transform can be identified with the Euclidean X-ray transform applied to a function supported in the unit ball, modulo changing the parameter of integration on each geodesic. There is an analogous relation for even AH metrics. An even AH metric induces what we call an *even structure* on $(\overline{M}, \partial\overline{M})$ subordinate to its smooth structure. This is a subatlas of the atlas defining the smooth structure, with the property that all the transition maps for the even structure are even diffeomorphisms. One can use the even structure to define a new smooth structure $(\overline{M}_e, \partial\overline{M}_e)$ on the topological manifold with boundary underlying $(\overline{M}, \partial\overline{M})$ by introducing $r = \rho^2$ as a new defining function. As outlined at the end of Section 4 of [FG12], when viewed relative to the smooth structure $(\overline{M}_e, \partial\overline{M}_e)$, the metric $g$ is projectively compact in the sense that its Levi-Civita connection is projectively equivalent to a connection $\widehat{\nabla}$ smooth up to the boundary, i.e. its geodesics agree up to parametrization with the geodesics of $\widehat{\nabla}$. The connection $\widehat{\nabla}$ need not be the Levi-Civita connection of a metric as happens on hyperbolic space, but the Uhlmann-Vasy local injectivity result applies also to the X-ray transform for smooth connections, so local injectivity for even AH metrics follows just by quoting [UV16].

If the AH metric $g$ is not even, one can still introduce an even structure and a corresponding $(\overline{M}_e, \partial\overline{M}_e)$ by introducing $r = \rho^2$ as a new defining function. But in this case the connection $\widehat{\nabla}$ is no longer smooth up to the boundary: its Christoffel symbols have expansions in $\sqrt{r}$. If $\partial_\rho h_\rho\big|_{\rho=0} \neq 0$ in (0.0.2), then the Christoffel symbols have $r^{-1/2}$ terms so $\widehat{\nabla}$ is not even continuous up to the boundary. If $\partial_\rho h_\rho\big|_{\rho=0} = 0$ but $(\partial_\rho)^3 h_\rho\big|_{\rho=0} \neq 0$, then $\widehat{\nabla}$ has $\sqrt{r}$ terms so it is continuous but not Lipschitz. Our assumption that $g$ is even modulo $O(\rho^5)$ guarantees that $\widehat{\nabla}$ is at least a $C^1$ connection.

In principle one could try to extend directly the proof in [UV16] to the case of a $C^1$ connection like $\widehat{\nabla}$. But the microlocal methods do not seem very well suited to such an analysis. Instead we argue by perturbation: $\widehat{\nabla}$ is a perturbation of a smooth connection, and the perturbation gets smaller the closer one gets to the boundary. For the quantitative control needed to carry this out, we need to use not only the local injectivity result of [UV16], but also the associated stability estimate. We briefly indicate how this goes, beginning by describing this stability estimate.

Let $\overline{\nabla}$ be a smooth connection on a manifold $\overline{M}_e$ of dimension at least 3, with strictly convex boundary given by $r = 0$, and $\widetilde{M}$ a closed manifold containing $\overline{M}_e$. The authors of [UV16] constructed a one-parameter family of "artificial boundaries" near a point $p \in \partial\overline{M}_e$ given by $x = -\eta$, where $x \in C^\infty(\widetilde{M})$ satisfies $x(p) = 0$ and $dx(p) = -dr(p)$ and $\eta > 0$, and showed injectivity of the X-ray transform $\overline{I}$ of $\overline{\nabla}$ restricted to geodesics in $\overline{M}_e$ entirely contained in $U_\eta := \{x \geq -\eta\} \cap \{r \geq 0\}$ (see Figure 1). The proof is based on the construction of a family of "microlocalized normal operators" $\overline{A}_{\chi,\eta,\sigma}$ each one of which is, roughly speaking, the conjugate by exponential weights of the average of $\overline{I}f$ over the set of such geodesics passing through a given point. Here $\sigma$ is the parameter in the exponential weight and $\chi$ is a cutoff function. They showed that for appropriately chosen $\chi$, the operator $\overline{A}_{\chi,\eta,\sigma}$ is an elliptic pseudodifferential operator in Melrose's scattering calculus which for sufficiently

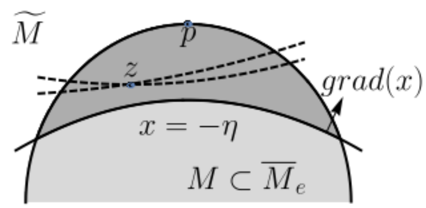


Figure 1: The artificial boundary.

small $\eta$ has trivial kernel when acting on functions supported in $U_\eta$, and derived the stability estimate

$$\|f\|_{L^2(U_\eta)} \leq C\|\overline{A}_{\chi,\eta,\sigma}f\|_{H^{1,0}_{sc}(O_\eta)}, \tag{0.0.4}$$

where $H^{1,0}_{sc}$ denotes a scattering Sobolev space and $O_\eta$ is a neighborhood of $U_\eta$ in $\overline{X}_\eta := \{x \geq -\eta\}$.

If $g$ is an AH metric even mod $O(\rho^N)$, its Levi-Civita connection is projectively equivalent as described above to a connection $\widehat{\nabla}$ of the form $\widehat{\nabla} = \overline{\nabla} + r^{N/2-1}B$ on $\overline{M}_e$, where $\overline{\nabla}$ and $B$ are smooth. If $N \geq 5$, then $\widehat{\nabla}$ is $C^1$, so the constructions of its X-ray transform $\widehat{I}$ and the operator $\widehat{A}_{\chi,\eta,\sigma}$ can be carried out just as for the smooth connection $\overline{\nabla}$. We show that the norm of the perturbation operator

$$\widehat{A}_{\chi,\eta,\sigma} - \overline{A}_{\chi,\eta,\sigma} : L^2(U_\eta) \to H^{1,0}_{sc}(O_\eta) \tag{0.0.5}$$

goes to zero as $\eta \to 0$. This gives an estimate of the form (0.0.4) for $\widehat{A}_{\chi,\eta,\sigma}$ for $\eta$ sufficiently small, which implies local injectivity since $\widehat{A}_{\chi,\eta,\sigma}$ factors through the X-ray transform $\widehat{I}$. The perturbation operator is estimated as in the classical Schur criterion bounding an $L^2$ operator norm by the supremum of the $L^1$ norms of the Schwartz kernel in each variable separately. We lift the kernels of the operators $\widehat{A}_{\chi,\eta,\sigma}$ and $\overline{A}_{\chi,\eta,\sigma}$ to a blown up space similar to Melrose's double stretched space (see [Mel94]), where their singularities are more easily analyzed. Due to the fact that the connection $\widehat{\nabla}$ is only of class $C^1$, some rather technical analysis is required near each boundary face and corner of the blow up to conclude that the kernel of $\widehat{A}_{\chi,\eta,\sigma}$ is sufficiently regular that the norm of the perturbation operator vanishes in the limit as $\eta \to 0$.

As is the case in [UV16], the method of the proof naturally yields reconstruction via a Neumann series and a stability estimate for $\widehat{I}$ acting between Sobolev spaces on $\overline{M}_e$ and on the sphere bundle $S^0\overline{M}_e$ of a smooth metric $g^0$ on $\overline{M}_e$, which we use to parametrize the geodesics of $\widehat{\nabla}$. One could pull back this estimate and obtain one for $I$ between function spaces on $M$ and on the sphere bundle for $g$ but the spaces so obtained are not natural, so we do not pursue this. Moreover, one could obtain a global injectivity result in exactly the same way as in [UV16] provided the compact manifold with boundary $\{\rho \geq \varepsilon\}$ admits a strictly convex foliation, for $\varepsilon$ sufficiently small. It would be of great interest to remove the assumption of evenness modulo $O(\rho^5)$ in Theorem 1; we anticipate that a different method would be necessary for addressing this. Also, the question of local injectivity for the X-ray transform in dimension 2 is a very interesting problem, which is open even in the setting of compact manifolds with boundary.

Chapter 2 is concerned with a question about the geometry of AH manifolds. Simple compact manifolds with boundary are a natural setting for the study the X-ray transform and, more generally, geometric inverse problems. Recall that a compact manifold with boundary is called simple if it is non-trapping, it has strictly convex boundary and no conjugate points. An analogous definition of a simple AH manifold was formulated in [GGS+] to address questions of tensor tomography and boundary rigidity. In the AH case, convexity of the boundary is in a sense automatic: for any boundary defining function $\rho$ and $\varepsilon > 0$ small enough the sets $\rho \geq \varepsilon$ are strictly convex. The definition in [GGS+] of a simple AH manifold is that the AH manifold be non-trapping (i.e. there exist no trapped geodesics) and without *boundary conjugate points*. Here absence of boundary conjugate points means by definition that there exists no pair of points $p^+$, $p^- \in \partial M$ and unit speed geodesic $\gamma$ with $\lim_{t\to\pm\infty}\gamma(t) = p^{\pm}$ such that there exists a non-trivial Jacobi field $Y$ along $\gamma$ satisfying $\lim_{t\to\pm\infty}|Y(t)|_g = 0$. It was shown in [GGS+] that these conditions imply that the geodesic flow is Anosov with respect to the Sasaki metric (see [Ebe73] for the definition), which together with the main result of [Kni18] implies that there are no conjugate points in the usual sense (*interior conjugate points*), i.e. there exists no non-trivial Jacobi field along any unit speed geodesic that vanishes for two distinct finite times. A result in [Ebe73] implies if an AH manifold has no interior conjugate points then there is no Jacobi field $Y(t)$ along a unit speed geodesic with the property $|Y(0)|_g = 0 = \lim_{t\to\infty} |Y(t)|_g$, that is, no "interior-boundary" conjugate points can occur. This raises the question of whether a non-trapping AH manifold without interior conjugate points necessarily does not exhibit boundary conjugate points, that is, whether all non-trapping AH manifolds without interior conjugate points are simple. The main result of Chapter 2, which is a joint work with Robin Graham, resolves this in the negative.

**Theorem 2.** *For any integer $n \geq 1$ there exist smooth non-trapping asymptotically hyperbolic manifolds of dimension $n+1$ with boundary conjugate points but no interior conjugate points.*

This result is related to work during the 1970s, when there was interest and activity concerned with understanding the relationships between various properties on a Riemannian manifold such as absence of conjugate points, Anosov geodesic flow, absence or presence of focal points, and existence of open sets of strictly positive curvature (see, for instance [Ebe73], [Kli74], [Gul75]). Our approach is inspired by techniques used in [Gul75] to construct metrics elucidating the relationships between some of these properties. Such questions remain of current interest; see, for example, §2.3 of [GLT] where methods of [Gul75] are used to construct an asymptotically conic metric on $\mathbb{R}^n$ which has positive curvature on an open set but no conjugate points.

Theorem 2 is proved by constructing explicit examples of manifolds having the properties stated. We start by constructing a non-trapping, complete, $O(n+1)$-invariant $C^{1,1}$ metric on $\mathbb{R}^{n+1}$ which compactifies to an AH metric, such that there are no nontrivial Jacobi fields that vanish twice in the interior but along radial geodesics there are Jacobi fields that vanish as both $t \to \pm\infty$. Here the $C^{1,1}$ regularity implies existence and uniqueness of geodesics; Jacobi fields are understood in a weak sense. Our manifold has constant positive sectional curvature in an open geodesic ball and negative sectional curvature outside a compact set; when $n = 1$, the negative sectional curvature is constant whereas when $n \geq 2$ this is not the case. For our purposes, the size of the set of positive curvature has to be carefully chosen: if it is too large, interior conjugate points occur, whereas if it is too small no boundary conjugate points occur; there is a critical size for which there exist boundary conjugate points but no interior ones. Because of this, our analysis is much more delicate than that of [Gul75], where the conditions are open. Somewhat surprisingly, it turns out that for our $C^{1,1}$ metric one can compute exact formulas for all geodesics, sectional curvatures and Jacobi fields even though the manifold has non-constant curvature outside any compact set for $n \geq 2$. For this reason our $C^{1,1}$ metric may be of more general interest.

In the second half of the chapter (Section 2.2) we show that our metric can be approximated by smooth metrics that still have all the required properties. As already hinted, these properties are quite unstable under perturbations of the metric: small variations can result in either presence of interior conjugate points or absence of boundary ones. The analogous approximation step in [Gul75] was trivial; any smooth, or even real-analytic, metric sufficiently close continued to satisfy the requisite conditions. We analyze the *stable* Jacobi fields, defined as those which vanish as $t \to \infty$. By careful choice of parameters in our approximating metric we arrange that there is a stable Jacobi field along radial geodesics which also vanishes as $t \to -\infty$ so that the corresponding metric has boundary conjugate points. We then derive a criterion (Proposition 2.2.18) in terms of the behavior of the stable solution for certain second order ODE's that rules out solutions vanishing twice. The relevant behavior can be controlled under perturbations of the metric to rule out interior conjugate points. Our argument requires control over third derivatives of the stable solutions (two in a parameter and one in the time variable) as the approximating metric approaches the $C^{1,1}$ metric, for which we have to carry out some rather technical analysis. The work included in Chapter 2 can also be found as a standalone publication ([EG]).

In Chapter 3 we remain in the global setting: we work on a simple AH manifold and address the question of stability of the geodesic X-ray transform. Roughly speaking, this means that our goal is to establish that small perturbations of the X-ray transform (the "measurement") cannot originate from large perturbations of the unknown function $f$, and find appropriate spaces to measure the "size" of a perturbation. In this chapter we make a slight change in notation; now an AH manifold will be denoted by $(\mathring{M}^{n+1}, g)$ and it will be the interior of a smooth compact manifold with boundary $M$. A generic smooth boundary defining function for $\partial M$ will be denoted by $x$.

Our approach to stability is mainly inspired by two works, namely those of Stefanov-Uhlmann [SU04] and of Berenstein-Casadio Tarabusi ([BC91]), both of which analyze the *normal operator* to the X-ray transform in different settings. On a simple compact manifold with boundary $(X, \widetilde{g})$, which is the setting of [SU04], the normal operator is given by $\mathcal{N}_{\widetilde{g}} = I^*I$, where

$$I^*F(z) = \int_{S_z^*X} F(\xi)d\mu_{\widetilde{g}}(\xi), \quad F \in C^\infty(S^*X),\ z \in X,$$

is the back-projection; here $d\mu_{\widetilde{g}}$ is the measure induced on each fiber of $S_z^*X$ by the Lebesgue measure on $T_z^*\mathring{M}$ and for $f \in C^\infty(X)$, $If$ is understood as a function on the unit cosphere bundle $S^*X := \{(z,\xi) \in T^*X : |\xi|_{\widetilde{g}} = 1\}$ which is constant along the orbits of the geodesic flow. For now the notation $I^*$ is formal, however $I^*$ can be interpreted as a formal adjoint for $I$ using suitable inner products and function spaces (this is discussed in Section 3.1 for the AH case). The authors of [SU04] showed that $\mathcal{N}_{\widetilde{g}}$ extends to an elliptic pseudodifferential operator of order $-1$ on $\widetilde{X}$, where $\widetilde{X}$ is an open domain slightly larger than $X$ and of the same dimension, such that its closure is still simple. The ellipticity of $\mathcal{N}_{\widetilde{g}}$ then implied the existence of a left parametrix (inverse up to compact error) that allowed them to obtain a stability estimate of the form

$$\|u\|_{L^2(X)} \leq C\|\mathcal{N}_{\widetilde{g}}u\|_{H^1(\widetilde{X})}, \quad u \in L^2(\widetilde{X}),\ \operatorname{supp} u \subset X, \tag{0.0.6}$$

using injectivity of $I$ on simple manifolds, which had already been established in the '70s. The construction of the normal operator carries over in the same way on hyperbolic space and it is well defined on $C^\infty$ functions of suitable decay at infinity; the authors of [BC91] derived explicit inversion formulas for it using the spherical Fourier transform for radial distributions on hyperbolic space (see [Hel99]). Even though they did not explicitly state a stability estimate, the estimate of Theorem 3 below for the special case of hyperbolic space follows immediately from their work using the machinery of the 0-calculus, which we will discuss shortly.

In [GGS$^+$] it was shown that simplicity of an AH manifold suffices to show that $I$ is injective on $xC^\infty(M)$ (in fact it is shown there that one can allow for trapped geodesics as well, provided that the trapped set is hyperbolic for the geodesic flow). The method of proof relied on showing that functions a priori in $xC^\infty(M)$ that lie in the nullspace of $I$ actually vanish to infinite order at $\partial M$ and it does not yield stability. The main result of Chapter 3 is a stability estimate analogous to (0.0.6) on simple AH manifolds and a strengthened injectivity result. The normal operator on a simple AH manifold $(\mathring{M}, g)$ is defined similarly to the case of simple compact manifolds with boundary: one lets

$$\mathcal{N}_g f = I^* I f(z) = \int_{S_z^* \mathring{M}} If(\xi) d\mu_g(\xi), \quad f \in \dot{C}^\infty(M),\ z \in \mathring{M},$$

where $\dot{C}^\infty(M)$ denotes smooth functions vanishing to infinite order at the boundary and $d\mu_g$ is the measure induced on the fibers of $S^*\mathring{M}$ by $g$, as before. In our setting $\mathcal{N}_g$ turns out to be a relatively well behaved object that can be studied within the framework of the 0-calculus of pseudodifferential operators of Mazzeo and Melrose. Those were introduced in [MM87] and further developed in [Maz91] to study differential and pseudodifferential operators on asymptotically hyperbolic manifolds (among other spaces in the case of [Maz91]), also see [Lau03]. 0-pseudodifferential operators generalize the *uniformly degenerate 0-differential operators*, consisting of the enveloping algebra of *0-vector fields*: those are the smooth vector fields on $M$ that vanish on $\partial M$ and are denoted by $\mathcal{V}_0$. They can be written locally near $\partial M$ as smooth linear combinations of $\{x\partial_x, x\partial_{y^1}, \dots, x\partial_{y^n}\}$, where $y^\alpha$ restrict to coordinates on $\partial M$. Our stability estimate will be in terms of certain weighted Sobolev spaces on which 0-pseudodifferential operators naturally act: we let $dV_g$ be the Riemannian volume density on $M$ induced by $g$ and for $k \in \mathbb{N}_0 = \{0, 1, \dots\}$ we let

$$x^\delta H_0^k(M; dV_g) = \{u \in x^\delta L^2(M; dV_g) : x^{-\delta} V_1 \cdots V_m u \in L^2(M; dV_g),\ m \le k,\ V_j \in \mathcal{V}_0\}.$$

If $s \geq 0$ then $H_0^s(M; dV_g)$ is defined by interpolation and for $s < 0$ by duality with respect to the $L^2(M; dV_g)$ pairing. Fixing vector fields $V_j \in \mathcal{V}_0$ in coordinate patches we can make sense of the norms $\| \cdot \|_{x^\delta H_0^k(M; dV_g)}$. As we will show in Section 3.3, it turns out that $I$ and $\mathcal{N}_g$ can be extended to operators on $x^\delta L^2(M; dV_g)$ for $\delta > -n/2$, bounded into appropriate weighted Sobolev spaces; specifically for $\mathcal{N}_g$ we have that it is bounded $x^\delta L^2(M; dV_g) \to x^{\delta'} H_0^1(M; dV_g)$ provided $\delta' \leq \delta$, $\delta > -n/2$ and $\delta' < n/2$. The main result of this chapter is the following:

**Theorem 3.** *Let $(\mathring{M}^{n+1}, g)$ be a simple AH manifold, $n \geq 1$. Then $I$ and $\mathcal{N}_g = I^* I$ are injective on $x^\delta L^2(M; dV_g)$, $\delta > -n/2$. Moreover, one has the stability estimate:*

$$\|u\|_{x^\delta H_0^s(M; dV_g)} \leq C \|\mathcal{N}_g \, u\|_{x^\delta H_0^{s+1}(M; dV_g)}, \quad \delta \in (-n/2, n/2), \; s \geq 0.$$

Note that $x C^\infty(M) \subset x^\delta L^2(M; dV_g)$ provided $\delta < 1 - n/2$, so Theorem 3 includes the injectivity result of [GGS+] on simple AH manifolds as a special case. However, their result is used in an essential way in the proof, similarly to the way the injectivity of $I$ on simple compact manifolds with boundary was used to derive (0.0.6) in [SU04].

As mentioned before, the proof of Theorem 3 uses the 0-calculus. As we show in Section 3.3, $\mathcal{N}_g$ is an elliptic pseudodifferential operator in $\Psi_0^{-1,n,n}(M)$ in the large 0-calculus (that is, it is a pseudodifferential operator of order $-1$ whose Schwartz kernel vanishes to order $n$ at the side faces of the 0-stretched product, see Section 3.2). Its *model operator*[3] can be identified with $\mathcal{N}_h$, where $h$ is the hyperbolic metric on the Poincaré ball; using the explicit inversion formulas for $\mathcal{N}_h$ derived in [BC91] and methods developed in [MM87] and [Maz91] we construct a left parametrix for $\mathcal{N}_g$. In [MM87] and [Maz91] parametrices were constructed for elliptic 0 and edge differential operators, whereas here we apply those techniques to construct a parametrix for a pseudodifferential operator. The parametrix is then used in two ways: firstly, one obtains an estimate

$$\|u\|_{x^\delta H_0^s(M; dV_g)} \leq C \left( \|\mathcal{N}_g u\|_{x^\delta H_0^{s+1}(M; dV_g)} + \|K u\|_{x^\delta H_0^s(M; dV_g)} \right), \quad \delta \in (-n/2, n/2), \; s \geq 0, \tag{0.0.7}$$

[3] The model operator is typically called the normal operator; however, we use this name to avoid confusion with the normal operator $\mathcal{N}_g$.

where $K : x^\delta H_0^s(M; dV_g) \to x^\delta H_0^s(M; dV_g)$ is a compact operator. Next, using the Mellin transform and the parametrix it can be shown that any function $u \in x^\delta L^2(M; dV_g)$ in the nullspace of $\mathcal{N}_g$, where $\delta > -n/2$, is smooth in $\mathring{M}$ and has a polyhomogeneous expansion at $\partial M$, vanishing there to order at least $n$. The author is indebted to Rafe Mazzeo for showing him this argument, which is similar in spirit to the constructions of polyhomogeneous expansions for elements in the nullspace of elliptic edge differential operators in Section 7 of [Maz91]. In [GGS$^+$, Proposition 3.15] it is shown that if $u \in xC^\infty(M)$ lies in the nullspace of $I$ then $u$ vanishes to infinite order on $\partial M$, and one checks that the proof also works for $u$ a priori assumed polyhomogeneous and vanishing to order at least 1 at $\partial M$. Since the nullspace of $\mathcal{N}_g$ agrees with that of $I$, it follows that $u$ is in the nullspace of the latter and polyhomogeneous, hence it vanishes to infinite order at $\partial M$. Once this has been established, the injectivity argument in [GGS$^+$] using Pestov identities applies to conclude that $u \equiv 0$. Finally the injectivity of $\mathcal{N}_g$ together with (0.0.7) yields Theorem 3 using a standard functional analysis result.

On compact manifolds with boundary, stability of the X-ray transform has been extensively studied; we mention several related results. Many of the works below also include stability results for the X-ray transform acting on tensor fields. On non-trapping manifolds with strictly convex boundary (*compact dissipative Riemannian manifolds*) that satisfy a curvature condition that excludes conjugate points (thus for a subclass of simple manifolds), a non-sharp stability estimate for $I$ is proved in [Sha94] (Theorem 4.3.3) citing earlier works of Mukhometov and Mukhometov-Romanov in the '70s ([Muk77], [MR78]), among others. As already mentioned, on simple compact manifolds with boundary, a stability estimate for the normal operator similar to the one in Theorem 3 is derived in [SU04]; also see [FSU08] for an analogous result for weighted X-ray transforms over general families of curves without conjugate points, defined in an appropriate sense. In [AS] a sharp stability estimate was obtained for $I$ on simple manifolds. In the presence of conjugate points, the result of Uhlmann and Vasy ([UV16]) shows stability on compact manifolds with strictly convex boundary of dimension at least 3 that satisfy a foliation condition by strictly convex hypersurfaces. Those manifolds can have conjugate points. Moreover, the results in [HU18] imply that under geometric assumptions stability holds on manifolds with conjugate points for certain Sobolev spaces. In the non-compact setting, sharp stability estimates are known for the X-ray transform in $\mathbb{R}^n$ (see [Nat86], Section II.5). As we already mentioned, in the case of hyperbolic space a stability estimate as in Theorem 3 follows immediately from the work of [BC91]; moreover, the inversion of the hyperbolic Radon transform on the two dimensional hyperbolic space has been numerically implemented in a stable manner ([LP97], [FKL$^+$00]). In the setting of AH manifolds, as already mentioned, the proof of the main result in Chapter 1 can be used to derive a stability estimate in the local setting, which can be made global if one assumes a foliation condition.

A future direction would be to use Theorem 3 to obtain a stability estimate in terms of $I$; based on the case of compact manifolds with boundary we expect that it is possible to show that an appropriately weighted $L^2$ norm of a function is estimated by a suitable Sobolev norm of its X-ray transform with order of regularity $1/2$, by showing a suitable mapping property for $I^*$. We plan to pursue this in the immediate future. It would also be interesting to explore whether stability still holds in the AH setting when one relaxes the simplicity assumption. In the compact manifold with boundary setting, presence of conjugate points in the interior of a compact Riemannian surface causes stability to fail in dimension 2 ([SU12], [MSU15]), and it is natural to expect an analogous behavior in the AH setting. However, in dimension 3 and higher, additional geometric assumptions can allow for stability even if there are conjugate points ([UV16], [HU18]) so it is likely that analogous results hold on AH manifolds. It would be especially interesting to investigate whether stability or instability holds in the presence of boundary conjugate points (for instance, in the setting of Chapter 2). It would also be interesting to study stability in the presence of trapped geodesics; as already mentioned, in the case when the trapped set is hyperbolic for the geodesic flow, injectivity of $I$ on $xC^\infty(M)$ is known by [GGS$^+$] and stability can be shown on compact manifolds with strictly convex boundary, no conjugate points and hyperbolic trapped set (see [Gui17]).

# Chapter 1

## LOCAL INJECTIVITY FOR THE GEODESIC X-RAY TRANSFORM ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

As already discussed in the Introduction, this chapter is concerned with local injectivity for the X-ray transform on asymptotically hyperbolic manifolds, which is proved via projective compactification of an AH manifold $(M, g)$. In Section 1.1 we define even structures on a manifold with boundary and construct the new manifold with boundary $(\overline{M}_e, \partial\overline{M}_e)$ obtained by introducing $r = \rho^2$ as a new defining function. We show that via this construction, even asymptotically hyperbolic metrics are the same as projectively compact metrics, only viewed relative to different smooth structures near infinity. In Section 1.2 we use this observation to relate the X-ray transforms for $g$ and $\widehat{\nabla}$, and then deduce Theorem 1 for even AH metrics. Section 1.3 begins the analysis for the $C^1$ connection $\widehat{\nabla}$ arising from an AH metric even mod $O(\rho^5)$. We decompose $\widehat{\nabla}$ into a smooth projectively compact connection $\overline{\nabla}$ and a nonsmooth error term and extend both to the larger manifold $\widetilde{M}$. We also prove Lemma 1.3.1, which states that the exponential map for $\widehat{\nabla}$ has one more degree of regularity than expected. In Section 1.4 we review scattering Sobolev spaces on a manifold with boundary, the construction of the microlocal normal operator $A_{\chi,\eta,\sigma}$ and the stability estimate (0.0.4), and show how Theorem 1 follows from Proposition 1.4.6, which is the assertion that the norm of the perturbation operator (0.0.5) goes to zero as $\eta \to 0$. In Section 1.5 we describe the blown-up double space, analyze in detail the lift of the kernel of $A_{\chi,\eta,\sigma}$ to this space, and conclude with the proof of Proposition 1.4.6. Throughout this chapter, lower case Latin indices $i$, $j$, $k$ label objects on $\overline{M}$ or $\overline{M}_e$ and run between 0 and $n$ in coordinates. Lower case Greek indices $\alpha$, $\beta$, $\gamma$ label objects on $\partial\overline{M} = \partial\overline{M}_e$ and run between 1 and $n$ in coordinates. So a Latin index corresponds to a pair $i \leftrightarrow (0, \alpha)$.

## 1.1 Even Asymptotically Hyperbolic = Projectively Compact

The proof of the main result of this chapter (Theorem 1) is based on an equivalence between even asymptotically hyperbolic metrics and projectively compact metrics, briefly outlined at the end of Section 4 of [FG12]. Since it is of central importance, we describe this equivalence in more detail. We begin by recalling the notions of projective equivalence and projectively compact metrics. A reference for projective equivalence is [Poo81, §5.24].

Two torsion-free connections $\nabla$ and $\widehat{\nabla}$ on a smooth manifold are said to be projectively equivalent if they have the same geodesics up to parametrization. This is equivalent to the condition that their difference tensor $\widehat{\nabla} - \nabla$ is of the form $v_{(i}\delta_{j)}^k = \frac{1}{2}(v_i\delta_j^k + v_j\delta_i^k)$ for some 1-form $v$. If $\gamma(t)$ is a geodesic for $\nabla$, then $\gamma(t(\tau))$ is a geodesic for $\widehat{\nabla}$, where $t(\tau)$ solves the differential equation $t'' = \mu(t)(t')^2$ with $\mu(t) = -v_{\gamma(t)}(\gamma'(t))$. If $v = du$ happens to be exact, then this equation for the parametrization reduces to the first order equation

$$t' = ce^{-u(\gamma(t))} \tag{1.1.1}$$

which can be integrated by separation of variables.

Let ${}^e g$ be a metric on the interior of a manifold with boundary $(\overline{M}_e, \partial\overline{M}_e)$. (The explanation for the super/subscript $e$ will be apparent shortly. For now this is just an inconsequential notation.) We say that ${}^e g$ is projectively compact if near $\partial\overline{M}_e$ it has the form

$${}^e g = \frac{dr^2}{4r^2} + \frac{k}{r},$$

where $r$ is a defining function for $\partial\overline{M}_e$ and $k$ is a smooth symmetric 2-tensor on $\overline{M}_e$ which is positive definite when restricted to $T\partial\overline{M}_e$. It is easily checked that this class of metrics is independent of the choice of defining function $r$. Elementary calculations (see (1.3.1) below) show that if ${}^e\nabla$ is the Levi-Civita connection of such a metric and $r$ a defining function, then the connection $\widehat{\nabla}$ defined by

$$\widehat{\nabla} = {}^e\nabla + D, \qquad D_{ij}^k = v_{(i}\delta_{j)}^k, \qquad v = dr/r \tag{1.1.2}$$

extends smoothly up to $\partial\overline{M}_e$. Thus ${}^e\nabla$ is projectively equivalent to the smooth connection $\widehat{\nabla}$. It turns out that projectively compact metrics are the same as even asymptotically hyperbolic metrics upon changing the smooth structure at the boundary. We digress to formulate the notion of an even structure on a manifold with boundary, which underlies this equivalence.

Set $\overline{\mathbb{R}^{n+1}_+} = \{(\rho, s) : \rho \geq 0, s \in \mathbb{R}^n\}$. View $\mathbb{R}^n \subset \overline{\mathbb{R}^{n+1}_+}$ as the subset $\rho = 0$.

**Definition 1.1.1.** *Let $\mathcal{U} \subset \overline{\mathbb{R}^{n+1}_+}$ be open. Let $f : \mathcal{U} \to \mathbb{R}$ be smooth. $f$ is said to be even (resp. odd) if either:*

1. *$\mathcal{U} \cap \mathbb{R}^n = \emptyset$, or*

2. *$\mathcal{U} \cap \mathbb{R}^n \neq \emptyset$ and the Taylor expansion of $f$ at each point of $\mathcal{U} \cap \mathbb{R}^n$ has only even (resp. odd) terms in $\rho$.*

It is equivalent to say that $f$ is even (resp. odd) if there is a smooth function $u$ so that $f(\rho, s) = u(\rho^2, s)$ (resp. $f(\rho, s) = \rho\, u(\rho^2, s)$). A smooth map $\varphi : \mathcal{U} \to \overline{\mathbb{R}^{n+1}_+}$ is said to be even if it is of the form $\varphi(\rho, s) = (\rho', s')$, where $\rho'$ is odd and each component of $s'$ is even.

**Definition 1.1.2.** *Let $(\overline{M}, \partial\overline{M})$ be a manifold with boundary, with atlas $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$. Let $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha\in\widetilde{\mathcal{A}}}$ be a subatlas of $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ corresponding to a subset $\widetilde{\mathcal{A}} \subset \mathcal{A}$. We say that $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha\in\widetilde{\mathcal{A}}}$ defines an even structure on $(\overline{M}, \partial\overline{M})$ subordinate to its smooth structure if the transition map*

$$\varphi_{\alpha_2} \circ \varphi_{\alpha_1}^{-1} : \varphi_{\alpha_1}(\mathcal{U}_{\alpha_1} \cap \mathcal{U}_{\alpha_2}) \to \varphi_{\alpha_2}(\mathcal{U}_{\alpha_1} \cap \mathcal{U}_{\alpha_2})$$

*is even for all $\alpha_1, \alpha_2 \in \widetilde{\mathcal{A}}$. The even structure is defined to be the maximal atlas containing $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha\in\widetilde{\mathcal{A}}}$ for which all transition maps are even.*

**Remark 1.1.3.** *Since $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha\in\widetilde{\mathcal{A}}}$ is in particular an atlas for the smooth structure determined by $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$, the even structure determines the smooth structure with respect to which it is subordinate. So there is really no need to begin with the original smooth structure. Nevertheless, we will usually have the smooth structure to start with and this language is* *appropriately suggestive. There are many different even structures subordinate to a given smooth structure.*

A diffeomorphism for some $\varepsilon > 0$ between a collar neighborhood of $\partial\overline{M}$ in $\overline{M}$ and $[0,\varepsilon) \times \partial\overline{M}$ induces an even structure on $(\overline{M}, \partial\overline{M})$. In fact, an atlas for $\partial\overline{M}$ induces an atlas for $[0,\varepsilon) \times \partial\overline{M}$ whose transition maps are the identity in the $\rho$ factor and independent of $\rho$ in the $\partial\overline{M}$ factor.

If $(\overline{M}, \partial\overline{M})$ is a manifold with boundary with subordinate even structure, it is invariantly defined to say that a function $f$ on $\overline{M}$ is even: $f \circ \varphi_\alpha^{-1}$ is required to be even on $\overline{\mathbb{R}^{n+1}_+}$ for all charts $(\mathcal{U}_\alpha, \varphi_\alpha)$ in the even structure. Likewise for odd functions. Conversely, knowledge of the even and odd functions on $(\overline{M}, \partial\overline{M})$ determines the subordinate even structure.

As an aside, we comment that if $(\overline{M}, \partial\overline{M})$ is a manifold with boundary, there is a natural one-to-one correspondence between smooth doubles of $(\overline{M}, \partial\overline{M})$ and subordinate even structures. Recall that a smooth double of $(\overline{M}, \partial\overline{M})$ is a choice of smooth manifold structure on the topological double $2\overline{M} = (\overline{M} \sqcup \overline{M})/\partial\overline{M}$ such that the inclusions $\overline{M} \to 2\overline{M}$ are diffeomorphisms onto their range and such that the natural reflection $2\overline{M} \to 2\overline{M}$ is a diffeomorphism. The even (resp. odd) functions on $(\overline{M}, \partial\overline{M})$ are determined by the double by the requirement that their reflection-invariant (resp. anti-invariant) extension to $2\overline{M}$ is smooth.

Denote by $S : \overline{\mathbb{R}^{n+1}_+} \to \overline{\mathbb{R}^{n+1}_+}$ the squaring map

$$S(\rho, s) = (\rho^2, s).$$

Let $(\overline{M}, \partial\overline{M})$ be a manifold with boundary and let $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha\in\widetilde{\mathcal{A}}}$ define an even structure on $(\overline{M}, \partial\overline{M})$ subordinate to its smooth structure. We construct another manifold with boundary $(\overline{M}_e, \partial\overline{M}_e)$ as follows. Set $\overline{M}_e = \overline{M}$ as topological spaces. Define

$$\psi_\alpha = S \circ \varphi_\alpha, \qquad \alpha \in \widetilde{\mathcal{A}}.$$

If $\alpha_1$, $\alpha_2 \in \widetilde{\mathcal{A}}$, then

$$(\varphi_{\alpha_2} \circ \varphi_{\alpha_1}^{-1})(\rho, s) = (\rho\, a(\rho, s), s'(\rho, s)),$$

where $a$ and the components of $s'$ are even. Now $\psi_{\alpha_2} \circ \psi_{\alpha_1}^{-1} = S \circ (\varphi_{\alpha_2} \circ \varphi_{\alpha_1}^{-1}) \circ S^{-1}$. Hence

$$\begin{aligned}(\psi_{\alpha_2} \circ \psi_{\alpha_1}^{-1})(r,s) =& \big(S \circ (\varphi_{\alpha_2} \circ \varphi_{\alpha_1}^{-1})\big)(\sqrt{r}, s)\\ =& S\big(\sqrt{r}\, a(\sqrt{r}, s), s'(\sqrt{r}, s)\big)\\ =& \big(ra(\sqrt{r}, s)^2, s'(\sqrt{r}, s)\big).\end{aligned}$$

Since $a$ and the components of $s'$ are even, it follows that $\psi_{\alpha_2} \circ \psi_{\alpha_1}^{-1}$ is smooth. Hence the charts $\{(\mathcal{U}_\alpha, \psi_\alpha)\}_{\alpha\in\widetilde{\mathcal{A}}}$ define a manifold with boundary structure on the topological space $\overline{M}$, which we denote $(\overline{M}_e, \partial\overline{M}_e)$. As topological spaces we have $\overline{M} = \overline{M}_e$. On the interior, the identity $\mathcal{I} : M \to M_e$ is a diffeomorphism. Since $\psi_\alpha \circ \varphi_\alpha^{-1} = S$ is smooth, it follows that $\mathcal{I} : \overline{M} \to \overline{M}_e$ is smooth. But $\mathcal{I}^{-1} : \overline{M}_e \to \overline{M}$ is not smooth since in the charts $\psi_\alpha$, $\varphi_\alpha$, its first component is the function $\sqrt{r}$ on $\overline{\mathbb{R}^{n+1}_+}$. The process of passing from $(\overline{M}, \partial\overline{M})$ with its subordinate even structure to $(\overline{M}_e, \partial\overline{M}_e)$ could be called "introducing $r = \rho^2$ as a new defining function".

Next consider the inverse process of "introducing $\rho = \sqrt{r}$ as a new defining function". Let $(\overline{N}, \partial\overline{N})$ be any manifold with boundary. We construct another manifold with boundary $(\overline{M}, \partial\overline{M})$ with subordinate even structure, such that $(\overline{N}, \partial\overline{N})$ equals $(\overline{M}_e, \partial\overline{M}_e)$ as manifolds with boundary. To do so, let $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\mathcal{A}}$ be an atlas for $(\overline{N}, \partial\overline{N})$. Take $\overline{M} = \overline{N}$ as topological spaces. Use as charts on $\overline{M}$ the maps $\varphi_\alpha = S^{-1} \circ \psi_\alpha$. Now

$$(\psi_{\alpha_2} \circ \psi_{\alpha_1}^{-1})(r,s) = (rb(r,s), s'(r,s))$$

where $b$ and $s'$ are smooth. Calculating the compositions as above gives

$$(\varphi_{\alpha_2} \circ \varphi_{\alpha_1}^{-1})(\rho,s) = \left(\rho\sqrt{b(\rho^2,s)}, s'(\rho^2,s)\right).$$

Since $b(0,s) \neq 0$, this is an even diffeomorphism. The atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ thus defines the desired manifold with boundary $(\overline{M}, \partial\overline{M})$ with subordinate even structure. In this case the subatlas $\widetilde{\mathcal{A}}$ equals $\mathcal{A}$.

Suppose now that $g$ is an AH metric on the interior $M$ of a compact manifold with boundary $(\overline{M}, \partial\overline{M})$ with a subordinate even structure. In the context of this discussion it is natural to define $g$ to be even relative to the chosen even structure if in coordinates $(\rho, s)$ in the even structure it has the form

$$g = \rho^{-2}\big(\overline{g}_{00}d\rho^2 + 2\overline{g}_{0\alpha}d\rho ds^\alpha + \overline{g}_{\alpha\beta}ds^\alpha ds^\beta\big) \tag{1.1.3}$$

with $\overline{g}_{00}$, $\overline{g}_{\alpha\beta}$ even and $\overline{g}_{0\alpha}$ odd. The choice of a representative $h$ for the conformal infinity induces a diffeomorphism between $[0,\varepsilon)\times\partial\overline{M}$ and a collar neighborhood of $\partial\overline{M}$ with respect to which $g$ has the form (0.0.2) with $h_0 = h$. By analyzing the construction of the normal form in [GL91], it is not hard to see that this diffeomorphism putting $g$ into normal form is even relative to the coordinates $(\rho, s)$ and the even structure determined by the product $[0,\varepsilon)\times\partial\overline{M}$ (see the proof of [Gui05, Lemma 2.1] for the special case when (1.1.3) is already in normal form relative to another representative). It follows that $g$ is even as defined in the introduction and that $g$ uniquely determines the even structure with respect to which it is even. In the other direction, an even AH metric in the sense of the introduction is clearly even with respect to the even structure determined by any of its normal forms. Thus an AH metric $g$ is even in the sense of the introduction if and only if it is even relative to some even structure subordinate to the smooth structure on $(\overline{M},\partial\overline{M})$, and this even structure is uniquely determined by $g$.

If $g$ is an even AH metric, we can consider the smooth manifold with boundary $(\overline{M}_e,\partial\overline{M}_e)$ obtained from the even structure determined by $g$ upon introducing $r=\rho^2$ as a new boundary defining function. Since $\mathcal{I}^{-1}: M_e \to M$ is a diffeomorphism, ${}^e g := (\mathcal{I}^{-1})^* g$ is a metric on $M_e$. We claim that ${}^e g$ is projectively compact relative to the smooth structure on $(\overline{M}_e,\partial\overline{M}_e)$. In fact, if $g$ has the form (0.0.2) on $[0,\varepsilon)\times\partial\overline{M}$ with $h_\rho$ even in $\rho$, then

$${}^e g = \frac{dr^2}{4r^2} + \frac{k_r}{r}, \tag{1.1.4}$$

where $k_r = h_{\sqrt{r}}$ is a one-parameter family of metrics on $\partial\overline{M}_e$ which is smooth in $r$. Thus ${}^e g$ is projectively compact. Conversely, a projectively compact metric relative to $(\overline{M}_e,\partial\overline{M}_e)$ is an even AH metric when viewed relative to $(\overline{M},\partial\overline{M})$.

In summary, the class of even asymptotically hyperbolic metrics on the interior of a manifold with boundary $(\overline{M},\partial\overline{M})$ with subordinate even structure is the same as the class of projectively compact metrics in the interior of $(\overline{M}_e, \partial\overline{M}_e)$. The distinction is just a matter of which smooth structure one chooses to use at infinity. The smooth structures are related by introducing $r = \rho^2$ as a new boundary defining function.

### *1.2 Local Injectivity for Even Metrics*

Let $(\overline{M}, \partial\overline{M})$ be a manifold with boundary and $g$ an even AH metric on $M$. As described in Section 1.1, the associated metric ${}^e g$ obtained by introducing $r = \rho^2$ as a new boundary defining function is projectively compact. In particular, for any defining function $r$ for $\partial\overline{M}_e$, the connection $\widehat{\nabla}$ defined by (1.1.2) is smooth up to $\partial\overline{M}_e$. We will reduce analysis of the local X-ray transform of $g$ to that for $\widehat{\nabla}$.

**Lemma 1.2.1.** *$\partial\overline{M}_e$ is strictly convex with respect to $\widehat{\nabla}$.*

*Proof.* Recall that this means that if $r$ is a defining function for $\partial\overline{M}_e$ with $r > 0$ in $M_e$ and if $\widehat{\gamma}$ is a nonconstant geodesic of $\widehat{\nabla}$ such that $r(\widehat{\gamma}(0)) = 0$ and $dr\big(\widehat{\gamma}\,'(0)\big) = 0$, then $\partial_\tau^2(r\circ\widehat{\gamma})|_{\tau=0} < 0$. Write $g$ in normal form (0.0.2) relative to a conformal representative $h$ on $\partial\overline{M}$, so that ${}^e g$ has the form (1.1.4) on $M_e$. Letting $\widehat{\Gamma}_{ij}^k$ (resp. ${}^e\Gamma_{ij}^k$) denote the Christoffel symbols of $\widehat{\nabla}$ (resp. the Christoffel symbols of the Levi-Civita connection ${}^e\nabla$ of ${}^e g$) an easy calculation (see (1.3.1) below) shows that ${}^e\Gamma_{\alpha\beta}^0 = 2k_{\alpha\beta} = 2h_{\alpha\beta}$ on $\partial\overline{M}_e$. Since $D_{\alpha\beta}^0 = 0$, we have at $\tau = 0$:

$$\partial_\tau^2(r\circ\widehat{\gamma}) = -\widehat{\Gamma}_{ij}^0\widehat{\gamma}^{i\prime}\widehat{\gamma}^{j\prime} = -\widehat{\Gamma}_{\alpha\beta}^0\widehat{\gamma}^{\alpha\prime}\widehat{\gamma}^{\beta\prime} = -{}^e\Gamma_{\alpha\beta}^0\widehat{\gamma}^{\alpha\prime}\widehat{\gamma}^{\beta\prime} = -2h_{\alpha\beta}\widehat{\gamma}^{\alpha\prime}\widehat{\gamma}^{\beta\prime} < 0.$$

□

It will be convenient to embed $\overline{M}_e$ in a smooth compact manifold without boundary $\widetilde{M}$ and to extend $\widehat{\nabla}$ to a smooth connection on $\widetilde{M}$, also denoted $\widehat{\nabla}$. If $\widehat{\gamma}$ is a geodesic of $\widehat{\nabla}$ with $\widehat{\gamma}(0) \in \overline{M}_e$, set $\tau_\pm(\widehat{\gamma}) := \pm\sup\{\tau \geq 0 : \widehat{\gamma}(t) \in \overline{M}_e \text{ for } 0 \leq \pm t \leq \tau\}$. If $U \subset \overline{M}_e$ (usually a small neighborhood of $p \in \partial\overline{M}$ or its closure), we define the set $\widehat{\Omega}_U$ of *$U$-local geodesics of*

$\widehat{\nabla}$ by

$$\widehat{\Omega}_U := \left\{\widehat{\gamma} : |\tau_\pm(\widehat{\gamma})| < \infty,\ \ |\tau_+(\widehat{\gamma})| + |\tau_-(\widehat{\gamma})| > 0,\ \ \widehat{\gamma}(t) \in U \text{ for } t \in [\tau_-(\widehat{\gamma}), \tau_+(\widehat{\gamma})]\right\}.$$

Here the requirement $|\tau_+(\widehat{\gamma})| + |\tau_-(\widehat{\gamma})| > 0$ excludes geodesics tangent to $\partial\overline{M}_e$.

If $f \in C(\overline{U})$, set

$$\widehat{I}f(\widehat{\gamma}) = \int_{\tau_-(\widehat{\gamma})}^{\tau_+(\widehat{\gamma})} f(\widehat{\gamma}(\tau))\, d\tau, \qquad \widehat{\gamma} \in \widehat{\Omega}_U. \tag{1.2.1}$$

The $U$-local X-ray transform of $f$ is the collection of all $\widehat{I}f(\widehat{\gamma})$, $\widehat{\gamma} \in \widehat{\Omega}_U$.

Recall that the parametrization of a geodesic of any connection on $TM_e$ is determined up to an affine change $\tau \to a\tau + b$, $a \neq 0$. Such a reparametrization changes $\widehat{I}f(\widehat{\gamma})$ by multiplication by $a^{-1}$. In particular, whether or not $\widehat{I}f(\widehat{\gamma}) = 0$ is independent of the parametrization. It suffices to restrict attention to geodesics whose parametrization satisfies a normalization condition. For instance, in the next section we fix a background metric $g^0$ and require that $|\widehat{\gamma}'(0)|_{g^0} = 1$.

Next we relate $I$ and $\widehat{I}$. This involves relating objects on $M$ with objects on $M_e$. Since $\mathcal{I} : M \to M_e$ is the identity map, this amounts to viewing the same object in a different smooth structure, i.e. in different coordinates near the boundary. We suppress writing explicitly the compositions with the charts $\psi_\alpha$, $\varphi_\alpha$. So the expression of the identity in these coordinates is $\mathcal{I}(\rho, s) = (\rho^2, s)$. Likewise, $g$ and ${}^e g$ are related in coordinates by setting $r = \rho^2$, as in (1.1.4). If $f$ is a function defined on $M$, we can regard $f$ as a function $f_e$ on $M_e$, related in coordinates by $f(\rho, s) = f_e(\rho^2, s)$. If $U \subset M$, set $U_e = \mathcal{I}(U)$.

If $\gamma(t)$ is a $U$-local geodesic for $g$, it is also a geodesic for ${}^e g$. Since ${}^e\nabla$ is projectively equivalent to $\widehat{\nabla}$, (1.1.1) and (1.1.2) imply that $\widehat{\gamma}(\tau) := \gamma(t(\tau))$ is a geodesic for $\widehat{\nabla}$, where $dt/d\tau = c\big(r(\gamma(t(\tau))\big)^{-1}$. Different choices of $c$ determine different parametrizations; imposition of a normalization condition on the parametrization as mentioned above provides one way to specify $c$ for each geodesic. The relation between $I$ and $\widehat{I}$ follows easily:

$$If(\gamma) = \int_{-\infty}^{\infty} f(\gamma(t))\, dt = c\int_{\tau_-(\widehat{\gamma})}^{\tau_+(\widehat{\gamma})} (r^{-1}f_e)(\gamma(t(\tau)))\, d\tau = c\widehat{I}(r^{-1}f_e)(\widehat{\gamma}). \tag{1.2.2}$$

Section 3.4 of [UV16] shows that if $U_e$ is a sufficiently small open neighborhood of $p \in \partial\overline{M}_e$, then the $U_e$-local X-ray transform for a smooth metric extends to a bounded operator on $L^2(U_e)$ with target space $L^2$ of a parametrization of the space of $U_e$-local geodesics with respect to a suitable measure. The same argument holds in our setting for a smooth connection such as $\widehat{\nabla}$. We will not make explicit the target $L^2$ space since we are only concerned here with injectivity.

Equation (1.2.2) shows that it is important to understand when $r^{-1}f_e \in L^2(U_e)$. Making the change of variable $r = \rho^2$ in the integral gives

$$\int (r^{-1}f_e)^2\, drds = 2\int (\rho^{-2}f)^2\rho\, d\rho ds = 2\int (\rho^{-3/2}f)^2\, d\rho ds.$$

Thus $r^{-1}f_e \in L^2(U_e, drds)$ if and only if $f \in \rho^{3/2}L^2(U, d\rho ds)$. In particular, $If(\gamma) = c\widehat{I}(r^{-1}f_e)(\widehat{\gamma})$ provides a definition of $If$ for $f \in \rho^{3/2}L^2(U, dv_{\overline{g}})$ consistent with its usual definition.

The main result of [UV16] is local injectivity of the geodesic X-ray transform for a smooth metric on a manifold with strictly convex boundary of dimension at least 3. However, the proof applies just as well for the X-ray transform for a smooth connection such as $\widehat{\nabla}$. In particular, the construction in the main text of the cutoff function $\chi$ for which the boundary principal symbol is elliptic is also valid for a connection since the right-hand side of the geodesic equation $\gamma^{k\,\prime\prime} = -\Gamma^k_{ij}\gamma^{i\prime}\gamma^{j\prime}$ is a quadratic polynomial in $\gamma'$. We do not need the extension of Zhou discussed in the appendix of [UV16], although that more general result applies as well. The main result of [UV16] transferred to our setting is as follows.

**Theorem 1.2.2.** *[UV16] Suppose that $\overline{M}_e$ has dimension at least 3 and let $p \in \partial\overline{M}_e$. Every neighborhood of $p$ in $\overline{M}_e$ contains a neighborhood $U_e$ of $p$ so that the $U_e$-local X-ray transform of $\widehat{\nabla}$ is injective on $L^2(U_e)$.*

*Proof of Theorem 1 for $g$ even.* The relation (1.2.2) shows that $f \in \rho^{3/2}L^2(U, dv_{\overline{g}})$ is in the kernel of the $U$-local transform for $g$ if and only if $r^{-1}f_e \in L^2(U_e)$ is in the kernel of the $U_e$-local transform for $\widehat{\nabla}$. Thus for $g$ even, Theorem 1 follows immediately from Theorem 1.2.2.

□

### 1.3 Connections Associated to AH Metrics Even mod $O(\rho^N)$

If the AH metric $g$ in (0.0.2) is not even, then the even structure on $(\overline{M}, \partial\overline{M})$ determined by a normal form for $g$ depends on the choice of normal form. We fix one such normal form and thus the even structure which it determines. We then construct $(\overline{M}_e, \partial\overline{M}_e)$ as above by introducing $r = \rho^2$ as a new boundary defining function. The metric ${}^e g$ would be projectively compact except that the corresponding one-parameter family $k_r = h_{\sqrt{r}}$ in (1.1.4) is no longer smooth: it has an expansion in powers of $\sqrt{r}$. The connection $\widehat{\nabla}$ defined by (1.1.2) involves first derivatives of $k_r$. As already discussed in the Introduction, assuming that $g$ is even mod $O(\rho^5)$ suffices to guarantee that $\widehat{\nabla}$ is Lipschitz continuous, and, in fact, that it extends to be $C^1$ up to $\partial\overline{M}_e$, though not necessarily $C^2$. However, near $\partial\overline{M}_e$, $\widehat{\nabla}$ can be viewed as a perturbation of a smooth connection $\overline{\nabla}$, which we proceed to describe.

Straightforward calculation from (1.1.4) shows that the Christoffel symbols of the connection $\widehat{\nabla}$ defined by (1.1.2) are given by:

$$\begin{aligned} \widehat{\Gamma}^0_{ij} &= \begin{pmatrix} 0 & 0 \\ 0 & 2(k_{\alpha\beta} - r\partial_r k_{\alpha\beta}) \end{pmatrix} \\ \widehat{\Gamma}^\gamma_{ij} &= \begin{pmatrix} 0 & \frac{1}{2}k^{\gamma\delta}\partial_r k_{\delta\beta} \\ \frac{1}{2}k^{\gamma\delta}\partial_r k_{\alpha\delta} & \Gamma^\gamma_{\alpha\beta} \end{pmatrix}, \end{aligned} \tag{1.3.1}$$

where $\Gamma^\gamma_{\alpha\beta}$ denotes the Christoffel symbols of $k_r$ with $r$ fixed. If $g$ is even mod $O(r^N)$ with $N$ odd, then $k = k^{(1)} + r^{N/2}k^{(2)}$ with $k^{(1)}$, $k^{(2)}$ smooth. It follows that all $\widehat{\Gamma}^k_{ij}$ have the form

$$\widehat{\Gamma}^k_{ij} = \overline{\Gamma}^k_{ij} + r^{N/2-1}B^k_{ij}$$

with $\overline{\Gamma}^k_{ij}$, $B^k_{ij}$ smooth up to $\partial\overline{M}_e$. Denote by $\overline{\nabla}$ the smooth connection with Christoffel symbols $\overline{\Gamma}^k_{ij}$. Recall that we have chosen a closed manifold $\widetilde{M}$ containing $\overline{M}_e$. Choose some smooth extension of $\overline{\nabla}$ to a neighborhood of $\overline{M}_e$, also denoted $\overline{\nabla}$. Then extend $\widehat{\Gamma}$ by

$$\widehat{\Gamma}^k_{ij} = \overline{\Gamma}^k_{ij} + r^{N/2-1}H(r)B^k_{ij} \tag{1.3.2}$$

where $H(r)$ is the Heaviside function. The extended connection $\widehat{\nabla}$ is then $C^{(N-3)/2}$.

It is an important fact for our analysis that the special structure of the connection $\widehat{\nabla}$ has as a consequence that its exponential map is more regular than one would expect. We consider the exponential map in the form $\widehat{\exp} : T\widetilde{M} \to \widetilde{M} \times \widetilde{M}$, defined by $\widehat{\exp}(z,v) = (z, \widehat{\varphi}(1,z,v))$, where $t \to \widehat{\varphi}(t,z,v)$ is the geodesic with $\widehat{\varphi}(0,z,v) = z$, $\widehat{\varphi}'(0,z,v) = v$. Since $\widehat{\nabla}$ is $C^{(N-3)/2}$ and $N \geq 5$, usual ODE theory implies that $\widehat{\exp}$ is a $C^{(N-3)/2}$ diffeomorphism from a neighborhood of the zero section onto its image. In fact, it has one more degree of differentiability. We formulate the result in terms of the inverse exponential map since that is how we will use it.

**Lemma 1.3.1.** *Let $\widehat{\nabla}$ be the $C^{(N-3)/2}$ connection defined by* (1.3.2)*, where $N \geq 5$ is an odd integer. Then $\widehat{\exp}^{-1}$ is $C^{(N-1)/2}$ in a neighborhood in $\widetilde{M} \times \widetilde{M}$ of the diagonal in $\partial\overline{M}_e \times \partial\overline{M}_e$.*

*Proof.* It suffices to show that $T\widetilde{M} \ni (z,v) \to \widehat{\varphi}(1,z,v) \in \widetilde{M}$ is $C^{(N-1)/2}$ near $(z,0)$ for $z \in \partial\overline{M}_e$. Work in coordinates $(r,s)$ for $z$ with respect to which ${}^e g$ is in normal form (1.1.4). Set $z = (z^0, z^\alpha) = (r, s^\alpha)$. For $v$ use induced coordinates $v = (v^0, v^\alpha)$ with $v = v^0\partial_r + v^\alpha\partial_{s^\alpha} = v^i\partial_{z^i}$ and set $w = (z,v)$. Write the flow as $\widehat{\varphi}(t,w) = (\widetilde{z}(t,w), \widetilde{v}(t,w))$. The geodesic equations are:

$$(\widetilde{z}^{\,k})' = \widetilde{v}^{\,k}, \qquad (\widetilde{v}^{\,k})' = -\widehat{\Gamma}^k_{ij}(\widetilde{z})\widetilde{v}^{\,i}\widetilde{v}^{\,j}. \tag{1.3.3}$$

Observe from (1.3.1) that all $\widehat{\Gamma}^k_{ij}$ are $C^{(N-1)/2}$ except for $\widehat{\Gamma}^\gamma_{0\alpha} = \widehat{\Gamma}^\gamma_{\alpha 0}$. So the right-hand sides of all equations in (1.3.3) are $C^{(N-1)/2}$ except for the equation for $(\widetilde{v}^{\,\gamma})'$. By (1.3.1), (1.3.2), this equation has the form

$$(\widetilde{v}^{\,\gamma})' = A^\gamma_{ij}(\widetilde{z})\widetilde{v}^{\,i}\widetilde{v}^{\,j} - 2\widetilde{r}^{N/2-1}H(\widetilde{r})B^\gamma_{0\beta}(\widetilde{z})\widetilde{v}^{\,0}\widetilde{v}^{\,\beta} \tag{1.3.4}$$

with $A^\gamma_{ij}$ of regularity $C^{(N-1)/2}$ and $B^\gamma_{0\beta}$ smooth. Using $\widetilde{r}' = \widetilde{v}^{\,0}$, write

$$\begin{aligned}
-2\widetilde{r}^{N/2-1}H(\widetilde{r})B^\gamma_{0\beta}(\widetilde{z})\widetilde{v}^{\,0}\widetilde{v}^{\,\beta} &= -\frac{4}{N}\Big(\widetilde{r}^{N/2}H(\widetilde{r})\Big)'B^\gamma_{0\beta}(\widetilde{z})\,\widetilde{v}^{\,\beta} \\
&= -\frac{4}{N}\Big(\widetilde{r}^{N/2}H(\widetilde{r})B^\gamma_{0\beta}(\widetilde{z})\,\widetilde{v}^{\,\beta}\Big)' \\
&\quad + \frac{4}{N}\widetilde{r}^{N/2}H(\widetilde{r})\Big(B^\gamma_{0\beta,k}(\widetilde{z})\,\widetilde{v}^{\,k}\widetilde{v}^{\,\beta} + B^\gamma_{0\beta}(\widetilde{z})\,(\widetilde{v}^{\,\beta})'\Big) \\
&= -\frac{4}{N}\Big(\widetilde{r}^{N/2}H(\widetilde{r})B^\gamma_{0\beta}(\widetilde{z})\,\widetilde{v}^{\,\beta}\Big)' + \frac{4}{N}\widetilde{r}^{N/2}H(\widetilde{r})C^\gamma_{ij}(\widetilde{z})\widetilde{v}^{\,i}\widetilde{v}^{\,j},
\end{aligned}$$

where for the last equality we have used (1.3.3) for $(\widetilde{v}^{\,\beta})'$, so that

$$C^{\gamma}_{ij}(\widetilde{z})\widetilde{v}^{\,i}\widetilde{v}^{\,j} = B^{\gamma}_{0\beta,k}(\widetilde{z})\,\widetilde{v}^{\,k}\widetilde{v}^{\,\beta} - B^{\gamma}_{0\beta}(\widetilde{z})\widehat{\Gamma}^{\beta}_{ij}(\widetilde{z})\widetilde{v}^{i}\widetilde{v}^{j}.$$

Note that $\widetilde{r}^{N/2}H(\widetilde{r})C^{\gamma}_{ij}(\widetilde{z})\widetilde{v}^{\,i}\widetilde{v}^{\,j}$ is $C^{(N-1)/2}$.

Therefore (1.3.4) can be rewritten in the form

$$\Big(v^{\gamma} + \frac{4}{N}\widetilde{r}^{N/2}H(\widetilde{r})B^{\gamma}_{0\beta}(\widetilde{z})\,\widetilde{v}^{\,\beta}\Big)' = \Big(A^{\gamma}_{ij}(\widetilde{z}) + \frac{4}{N}\widetilde{r}^{N/2}H(\widetilde{r})C^{\gamma}_{ij}(\widetilde{z})\Big)\widetilde{v}^{\,i}\widetilde{v}^{\,j}. \tag{1.3.5}$$

Now the linear transformation $\widetilde{v} \mapsto \widetilde{b} = L(\widetilde{z})\widetilde{v}$, where $\widetilde{b}^{\,\gamma} = \widetilde{v}^{\,\gamma} + \frac{4}{N}\widetilde{r}^{N/2}H(\widetilde{r})B^{\gamma}_{0\beta}(\widetilde{z})\,\widetilde{v}^{\,\beta}$, is of class $C^{(N-1)/2}$ in $(\widetilde{z},\widetilde{v})$ and is invertible for $\widetilde{r}$ small. Replacing (1.3.4) by (1.3.5) in (1.3.3) and setting $\widetilde{v} = L^{-1}(\widetilde{z})\widetilde{b}$ throughout, we obtain a system of ODE of the form

$$\Big(\widetilde{z},\widetilde{v}^{\,0},\widetilde{b}\Big)' = F\Big(\widetilde{z},\widetilde{v}^{\,0},\widetilde{b}\Big),$$

where $F$ is $C^{(N-1)/2}$. It follows that the map $(t,z,v) \mapsto \widehat{\varphi}(t,z,v)$ is of class $C^{(N-1)/2}$ upon setting $\widetilde{b}^{\,\gamma} = L^{\gamma}_{\beta}(\widetilde{z})\widetilde{v}^{\,\beta}$. □

Lemma 1.2.1 (the strict convexity of $\partial\overline{M}_e$) holds for both $\widehat{\nabla}$ and $\overline{\nabla}$ if $g$ is even mod $O(\rho^N)$ with $N \geq 5$ odd, with the same proof as before. We define the sets $\widehat{\Omega}_U$, $\overline{\Omega}_U$ of $U$-local geodesics for $\widehat{\nabla}$ and $\overline{\nabla}$ the same way as before. It will be useful to have a common parametrization for the sets of geodesics of $\widehat{\nabla}$ and $\overline{\nabla}$. For this purpose, fix a background metric $g^0$ on $\widetilde{M}$. Let $S^0\widetilde{M}$ denote the unit sphere bundle of $g^0$. For $v \in S^0\widetilde{M}$, denote by $\widehat{\gamma}_v$, (resp. $\overline{\gamma}_v$) the geodesic for $\widehat{\nabla}$ (resp. $\overline{\nabla}$) with initial vector $v$. We define the $U$-local X-ray transforms for $\widehat{\nabla}$ and $\overline{\nabla}$ just as in (1.2.1), except now we view them as functions on the subsets of $S^0\widetilde{M}$ corresponding to $\widehat{\Omega}_U$, $\overline{\Omega}_U$:

$$\widehat{I}f(v) = \int_{\tau_-(\widehat{\gamma}_v)}^{\tau_+(\widehat{\gamma}_v)} f(\widehat{\gamma}_v(\tau))\,d\tau$$

and similarly for $\overline{I}f(v)$. Sometimes we will use the notation $If(v)$ generically for $\widehat{I}f(v)$ or $\overline{I}f(v)$, or, for that matter, for the $U$-local X-ray transform for any $C^1$ connection on a manifold with strictly convex boundary. No confusion will arise with the notation $If(\gamma)$ from Section 1.2 for the X-ray transform for the AH metric $g$, since we will not be dealing with $g$ again except implicitly in the isolated instance where we deduce Theorem 1.

### *1.4 Stability and Perturbation Estimates*

We continue to work with the connections $\widehat{\nabla}$ and $\overline{\nabla}$ obtained from an AH metric even mod $O(\rho^N)$ with $N \geq 5$. From now on it will always be assumed that the dimension of the manifold $M$ is at least 3. Since $\overline{\nabla}$ is smooth and $\partial\overline{M}_e$ is strictly convex with respect to it, Theorem 1.2.2 (local injectivity) holds also for $\overline{\nabla}$. As mentioned in the Introduction, in order to deduce local injectivity for $\widehat{\nabla}$, the stability estimate derived in [UV16] for the microlocalized normal operator $\overline{A}_{\chi,\eta,\sigma}$ will be needed. This estimate is formulated in terms of scattering Sobolev spaces. In this section we review those spaces, the construction of the microlocalized normal operator, and the stability estimate proved in [UV16]. Then we formulate our main perturbation estimate (Proposition 1.4.6) and show how Theorem 1 follows from it. Proposition 1.4.6 will be proved in Section 1.5. In this section we work almost entirely on $\overline{M}_e$ and its extension $\widetilde{M}$ (with the exception of the very last proof), so we will not be using the subscript $e$ for its various subsets to avoid cluttering the notation.

We define polynomially weighted scattering Sobolev spaces on a compact manifold with boundary $(\overline{X}, \partial\overline{X})$ with $\dim \overline{X} = n+1$. Let $x$ be a boundary defining function for $\overline{X}$. The space of *scattering vector fields*, denoted by $\mathcal{V}_{sc}(\overline{X})$, consists of the smooth vector fields on $\overline{X}$ that are a product of $x$ and a smooth vector field tangent to $\partial\overline{X}$. This means that if $(x, y)$ are coordinates near $p \in \partial\overline{X}$, any element of $\mathcal{V}_{sc}(\overline{X})$ can be written as a linear combination over $C^\infty(\overline{X})$ of the vector fields $x^2\partial_x$, $x\partial_{y^\alpha}$, $\alpha = 1, \dots, n$. If $k \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$ define

$$H^{k,\beta}_{sc}(\overline{X}) = \{u \in x^\beta L^2(\overline{X}) : x^{-\beta} V_1 \dots V_m u \in L^2(\overline{X}) \text{ for } V_j \in \mathcal{V}_{sc} \text{ and } 0 \leq m \leq k\};$$

here $L^2$ is defined using a smooth measure on $\overline{X}$ [1]. Note that $H^{0,\beta}_{sc}(\overline{X}) = x^\beta L^2(\overline{X})$. For $s \geq 0$, $H^{s,\beta}_{sc}(\overline{X})$ can be defined by interpolation and for $s < 0$ by duality, though we will not need this. Upon fixing $N_0$ smooth vector fields $V_1, \dots, V_{N_0}$ such that any element of $\mathcal{V}_{sc}(\overline{X})$ can be written locally as a linear combination over $C^\infty(\overline{X})$ of a subset of them, one may

[1] Our notation slightly differs from that of [UV16] in that we use a smooth measure rather the scattering measure $x^{-(\dim \overline{X}+1)}dxdy$ to define our base $L^2$ space. The spaces here and in [UV16] are the same up to shifting the weight by $(\dim \overline{X} + 1)/2$.

define a norm on $H^{k,\beta}_{sc}(\overline{X})$ by letting

$$\|u\|^2_{H^{k,\beta}_{sc}(\overline{X})} = \sum_{m=0}^{k} \sum_{i_j \in \{1,\dots,N_0\}} \|x^{-\beta} V_{i_1} \dots V_{i_m} u\|^2_{L^2(\overline{X})}. \tag{1.4.1}$$

If $U \subset \overline{X}$ is open then $H^{k,\beta}_{sc}(U)$ will consist of distributions of the form $u\big|_U$, where $u \in H^{k,\beta}_{sc}\left(\overline{X}\right)$; the corresponding norm will be the same as (1.4.1) with the exception of replacing $\|\cdot\|_{L^2\left(\overline{X}\right)}$ by $\|\cdot\|_{L^2(U)}$.

We next review the arguments and results from [UV16] that we will need, starting with the construction of the artificial boundary mentioned in the Introduction.

**Lemma 1.4.1** ([UV16], Section 3.1)**.** *Let $p \in \partial\overline{M}_e$ and $\nabla$ be a $C^1$ connection with respect to which $\partial\overline{M}_e$ is strictly convex. There exists a smooth function $\hat{x}$ in a neighborhood $\mathcal{U}$ of $p$ in $\widetilde{M}$ with the properties:*

1. *$\hat{x}(p) = 0$*
2. *$d\hat{x}(p) = -dr(p)$*
3. *Setting $x_\eta := \hat{x} + \eta$, for any neighborhood $\widetilde{O}$ of $p$ in $\widetilde{M}$ there exists an $\eta_0$ such that $U_\eta := \{r \geq 0\} \cap \{x_\eta \geq 0\} \subset \widetilde{O}$ for $\eta \leq \eta_0$*
4. *For $\eta$ near 0 (positive or negative) the set $X_\eta := \{\hat{x} > -\eta\} = \{x_\eta > 0\} \subset \widetilde{M}$ has strictly concave boundary with respect to $\nabla$ locally near $p$.* [2]

Now let $g^0$ be the Riemannian metric on $\widetilde{M}$ chosen at the end of Section 1.3. By Property 2 in Lemma 1.4.1 we can assume that in $\mathcal{U}$ we have $d\hat{x} \not\equiv 0$. Hence for each sufficiently small $\eta$ (upon appropriately shrinking $\mathcal{U}$ if necessary) we can use the flow $\psi : \mathbb{R} \times \mathcal{U} \to \widetilde{M}$ of $\frac{\operatorname{grad} \hat{x}}{|\operatorname{grad} \hat{x}|^2_{g^0}} = \frac{\operatorname{grad} x_\eta}{|\operatorname{grad} x_\eta|^2_{g^0}}$ flowing from $\partial X_\eta$ to identify a collar neighborhood of $\partial X_\eta$ in $\overline{X}_\eta$ with $[0, \delta_0)_{x_\eta} \times \partial X_\eta$ for a small fixed $\delta_0$. By taking $0 \leq \eta < \delta_0$ we can always arrange that

[2] Recall that this means that for any $\nabla$-geodesic $\gamma(t)$ with $x_\eta(\gamma(0)) = 0$ and $dx_\eta(\gamma'(0)) = 0$ one has $\frac{d^2}{dt^2}\big|_{t=0} x_\eta \circ \gamma(t) > 0$.

$p \in [0,\delta_0)_{x_\eta} \times \partial X_\eta$. Upon shrinking $\partial X_\eta$ if necessary we can assume that $\psi(\eta,\cdot) : \partial X_\eta \to \partial X_0 =: Y_p$ is a diffeomorphism for all sufficiently small $\eta$. We use this diffeomorphism to finally identify a collar neighborhood of $\partial X_\eta$ in $\overline{X}_\eta$ with $[0,\delta_0)_{x_\eta} \times Y_p$. In terms of this identification, for fixed $\eta$ the metric $g^0$ takes the form $F dx_\eta^2 + g_{x_\eta}$ on $\overline{X}_\eta$ locally near the boundary, where $F = |dx_\eta|_{g^0}^{-2}$ is smooth and non-vanishing and for a fixed $\eta$, $g_{x_\eta}$ is a 1-parameter family of metrics on $Y_p$ (thus if one lets $\eta$ vary then $g_{x_\eta}$ depends on two parameters: $\eta$ fixes the manifold with boundary $(\overline{X}_\eta, \partial X_\eta)$ by determining the boundary defining function $x_\eta$, and then the value of $x_\eta$ at a given point determines the metric). Moreover, vectors in $S_z^0\widetilde{M}$, $z \in \overline{X}_\eta$, can be written in terms of the identification above as $v = \lambda\partial_{x_\eta} + \omega$, where $\omega \in TY_p$ (of course not necessarily of unit length, so our setup slightly differs from the one in [UV16], see Remark 1.4.4 below).

In order to show local injectivity of the X-ray transform, one needs a description of geodesics that stay within a given neighborhood:

**Lemma 1.4.2** ([UV16], Section 3.2)**.** *Let $\nabla$ be a $C^1$ connection with respect to which $\overline{M}_e$ has strictly convex boundary. There exist constants $\widetilde{C} > 0$, $0 < \delta_1 < \delta_2$, $c_0 > 0$ and $\eta_0 > 0$, and neighborhood $Z_p$ of $p$ in $Y_p$, such that if $0 \le \eta < \eta_0$ and if $\gamma(t)$ is a $\nabla$-geodesic with initial position $z = (x,y) \in [0,c_0]_{x_\eta} \times \overline{Z}_p \subset \overline{X}_\eta$ and velocity $v = (\lambda,\omega) \in S_z^0\widetilde{M}$ satisfying*

$$\left|\frac{\lambda}{|\omega|_{g^0}}\right| \le \widetilde{C}\sqrt{x} \tag{1.4.2}$$

*then one has $x_\eta \circ \gamma(t) \ge 0$ for $|t| \le \delta_2$ and $x_\eta \circ \gamma(t) \ge c_0$ for $|t| \ge \delta_1$.*

By taking $\eta_0 << c_0$ in Lemma 1.4.2 and by Lemma 1.4.1 one can always assume that a neighborhood of $U_\eta$ in $\overline{X}_\eta$ is contained in $[0,c_0]_{x_\eta} \times \overline{Z}_p$, and we will henceforth assume that this is the case. Now let $\delta_1$ and $\nabla$ be as in Lemma 1.4.2 and let $\exp : T\widetilde{M} \to \widetilde{M}$ be the exponential map of $\nabla$. If $v \in S_z^0\widetilde{M}$ satisfies the assumptions of the lemma and $f$ is continuous and supported in $[0,c_0)_{x_\eta} \times Z_p$, we have $If(v) = \int_{-\delta_1}^{\delta_1} f(\exp(tv))dt$, so for all such $v$ and $f$ one can define the X-ray transform by integrating only over a fixed finite interval. The authors of [UV16] consider $If$ only on vectors $v = (\lambda,\omega) \in S_z^0\widetilde{M}$ satisfying a stronger condition, namely that for some positive constant $C_2$ one has $\frac{|\lambda|}{|\omega|_{g^0}} \leq C_2 x$ with $z = (x, y) \in [0, c_0]_{x_\eta} \times \overline{Z}_p$ for $\eta$ sufficiently small, and construct a microlocalized normal operator for $I$. Specifically, with $f$ as before and $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \geq 0$ and $\chi(0) = 1$, let

$$A_{\chi,\eta} f(z) := \int_{S_z^0 \widetilde{M}} \chi\left(\frac{\lambda}{|\omega|_{g^0} x}\right) If(v) d\mu_{g^0}, \quad z = (x, y) \in [0, c_0]_{x_\eta} \times \overline{Z}_p, \tag{1.4.3}$$

where $d\mu_{g^0}$ is the measure induced on $S_z^0 \widetilde{M}$ by $g^0|_{T_z \widetilde{M}}$. Note that for any $C_2$, $c_0$ can be chosen sufficiently small that (1.4.2) is automatically satisfied in $[0, c_0]_{x_\eta} \times \overline{Z}_p$. The constant $C_2$ is fixed when $\chi \in C_c^\infty(\mathbb{R})$ is chosen (see Proposition 1.4.3 below), and then $c_0$, $\eta_0$ can be chosen so that the integrand in (1.4.3) is only supported on vectors corresponding to geodesics in $\Omega_{U_\eta}$. We also mention that for such choices of constants it follows from the proof of Lemma 1.4.2 that if $\gamma$ is a geodesic with initial position $z = (x, y) \in [0, c_0]_{x_\eta} \times \overline{Z}_p$ and initial velocity $v = (\lambda, \omega)$ such that $\chi\Big(\frac{\lambda}{|\omega|_{g^0} x}\Big) \neq 0$, then there exists $C > 0$ such that for any $|t| \leq \delta_2$ and $0 \leq \eta < \eta_0$

$$x_\eta \circ \gamma(t) \geq x - C^2 x^2. \tag{1.4.4}$$

Finally for $\sigma > 0$ define the conjugated microlocalized normal operator:

$$A_{\chi,\eta,\sigma} := x_\eta^{-2} e^{-\sigma/x_\eta} A_{\chi,\eta} e^{\sigma/x_\eta}.$$

We denote this operator in case $\nabla = \overline{\nabla}$ (resp. $\widehat{\nabla}$) by $\overline{A}_{\chi,\eta,\sigma}$ (resp. $\widehat{A}_{\chi,\eta,\sigma}$). In the case of the smooth connection $\overline{\nabla}$ on $\overline{M}_e$, for which $\partial \overline{M}_e$ is strictly convex, and in dimension $\geq 3$, it was proved in [UV16, Proposition 3.3] that $\overline{A}_{\chi,\eta,\sigma}$ are scattering pseudodifferential operators (in the notation there, $\overline{A}_{\chi,\eta,\sigma} \in \Psi_{sc}^{-1,0}(\overline{X}_\eta)$). This implies that they also act on scattering Sobolev spaces. The following Proposition contains the stability estimate we will need in terms of such spaces. We set $\psi_t(\cdot) = \psi(t, \cdot) : \mathcal{U} \to \widetilde{M}$.

**Proposition 1.4.3** ([UV16], Section 3.7)**.** *Suppose as before that $n + 1 \geq 3$ and let $\sigma > 0$. There exist $\chi_0 \in C_c^\infty(\mathbb{R})$, $\chi_0 \geq 0$, $\chi_0(0) = 1$, such that for any sufficiently small neighborhood $O$ of $p \in \partial \overline{M}_e$ in $\overline{X}$ there exist $\eta_0 > 0$ and $C_0 > 0$ with the property that for $0 \leq \eta \leq \eta_0$ one has $U_\eta \subset O_\eta := \psi_{-\eta}(O) \subset \overline{X}_\eta$, and the estimate*

$$\|u\|_{x^\beta L^2(U_\eta)} \leq C_0 \|\overline{A}_{\chi_0,\eta,\sigma} u\|_{H_{sc}^{1,\beta}(O_\eta)}, \quad \beta \in \mathbb{R}, \tag{1.4.5}$$

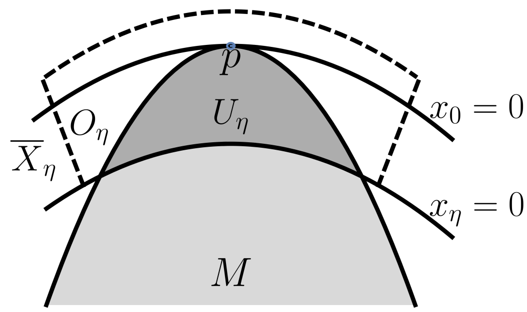


Figure 1.1: The level sets of $\hat{x}$.

*where $u \in x^\beta L^2(U_\eta)$ is extended by 0 outside $U_\eta$. Here the Sobolev spaces on subsets of $\overline{X}_\eta$ are defined by pulling back by $\psi_\eta$ the corresponding spaces on subsets of $\overline{X}_0$.*

**Remark 1.4.4.** *The estimate stated in [UV16, Section 3.7] is of the form*

$$\|u\|_{H^{s,\beta}_{sc}(\overline{X}_\eta)} \leq C_0 \|\overline{A}_{\chi_0,\eta,\sigma} u\|_{H^{s+1,\beta}_{sc}(\overline{X}_\eta)}, \quad s \geq 0, \ \operatorname{supp} u \subset U_\eta \tag{1.4.6}$$

*For $s = 0$ the space on the left is exactly $x^\beta L^2(U_\eta)$. On the other hand, the upper bound in* (1.4.6) *can be replaced by the one in* (1.4.5) *provided* $\operatorname{supp} u \subset U_\eta$*, since the Schwartz kernel of the operators $\overline{A}_{\chi_0,\eta,\sigma}$ has been localized in both factors near $U_\eta$, see for instance [UV16, Remark 3.2].*

*We also remark that the way we construct the operators $A_{\chi,\eta}$ differs slightly from the setup of [UV16], since we parametrize geodesics by their initial velocities normalized so that they have unit length with respect to the metric $g^0$, and average the transform over them using the measure induced by $g^0$ on the fibers of $S^0\widetilde{M}$. In [UV16] the geodesics are parametrized by writing their initial velocities as $(\lambda, \omega) \in \mathbb{R} \times \mathbb{S}^{n-1}$ using coordinates, and the measure used is $d\lambda d\omega$, where $d\omega$ is the standard measure on the unit sphere $\mathbb{S}^{n-1}$. However this difference doesn't affect the analysis, as already remarked there (see Remark 3.1 and the proof of Proposition 3.3).*

**Remark 1.4.5.** *As remarked in [UV16, Lemma 3.6], Proposition 1.4.3 holds for any* $\chi_0$ *sufficiently close to a specific Gaussian in the topology of Schwartz space. In particular,* $\chi_0$ *can be taken to be even, and from now on we assume that this is the case, since this simplifies the notation.*

Let $\chi_0$ be as in Proposition 1.4.3, chosen to be even. Let $\sigma > 0$ be fixed. Define

$$E_{\eta,\sigma} := \overline{A}_{\chi_0,\eta,\sigma} - \widehat{A}_{\chi_0,\eta,\sigma}$$

In Section 1.5.2 we will prove the following key proposition:

**Proposition 1.4.6.** *Let* $\sigma > 0$*. Provided* $O$ *is a sufficiently small neighborhood of* $p \in \partial M_e$ *in* $\overline{X}_0$*, for each* $\delta > 0$ *there exits* $\eta_0 > 0$ *with the property that if* $0 \leq \eta < \eta_0$ *one has* $U_\eta \subset\subset O_\eta := \psi_{-\eta}(O)$ *and*

$$\|E_{\eta,\sigma} u\|_{H^{1,0}_{sc}(O_\eta)} \leq \delta \|u\|_{L^2(U_\eta)}$$

*for all* $u \in L^2(U_\eta)$ *extended by 0 outside* $U_\eta$.

**Remark 1.4.7.** *In Proposition 1.4.6 one does not need to assume that* $n + 1 \geq 3$*, however unless this is the case Proposition 1.4.3 does not hold and the proof of Corollary 1.4.8 below breaks.*

An immediate consequence of Proposition 1.4.6 is the following:

**Corollary 1.4.8.** *With notations as before and assuming that* $\dim(\overline{M}_e) \geq 3$*, there exists* $\eta_0 > 0$ *such that for* $0 < \eta < \eta_0$ *the transform* $f \mapsto \widehat{I} f\big|_{\widehat{\Omega}_{U_\eta}}$ *is injective on* $L^2(U_\eta)$.

*Proof.* Fix $\sigma > 0$ and let $\chi_0$ be as in Proposition 1.4.3, even. Then take $O$ sufficiently small, as in Propositions 1.4.3 and 1.4.6, and let $C_0$ and $\eta_0$ be according to the former, corresponding to $O$. By Proposition 1.4.6, upon shrinking $\eta_0$ if necessary, for $0 \leq \eta < \eta_0$ we have

$$\|E_{\eta,\sigma} u\|_{H^{1,0}_{sc}(O_\eta)} \leq 1/(2C_0) \|u\|_{L^2(U_\eta)}$$

for $u \in L^2(U_\eta)$ extended by 0 elsewhere. Since $\overline{A}_{\chi_0,\eta,\sigma} = \widehat{A}_{\chi_0,\eta,\sigma} + E_{\eta,\sigma}$, if $u \in L^2(U_\eta)$ one has, for $0 \leq \eta < \eta_0$

$$\begin{aligned}\|u\|_{L^2(U_\eta)} \leq C_0\|\overline{A}_{\chi_0,\eta,\sigma}u\|_{H^{1,0}_{sc}(O_\eta)} &\leq C_0\|\widehat{A}_{\chi_0,\eta,\sigma}u\|_{H^{1,0}_{sc}(O_\eta)} + C_0\|E_{\eta,\sigma}u\|_{H^{1,0}_{sc}(O_\eta)} \\ &\leq C_0\|\widehat{A}_{\chi_0,\eta,\sigma}u\|_{H^{1,0}_{sc}(O_\eta)} + 1/2\|u\|_{L^2(U_\eta)} \Rightarrow \|u\|_{L^2(U_\eta)} \leq 2C_0\|\widehat{A}_{\chi_0,\eta,\sigma}u\|_{H^{1,0}_{sc}(O_\eta)}.\end{aligned}$$

This implies injectivity of $\widehat{A}_{\chi_0,\eta,\sigma}$ on $L^2(U_\eta)$. Using the definition of $\widehat{A}_{\chi_0,\eta,\sigma}$, the local X-ray transform $f \mapsto \widehat{I}f|_{\widehat{\Omega}_{U_\eta}}$ is injective on $e^{\sigma/x_\eta}L^2(U_\eta) \supset L^2(U_\eta)$. □

*Proof of Theorem 1.* The proof presented in Section 1.2 for the even case applies here verbatim, with the only difference that injectivity of the $U_e$-local transform for $\widehat{\overline{\nabla}}$ on $L^2(U_e)$ now follows from Corollary 1.4.8. □

## 1.5 Analysis of Kernels

The goal of this section is to prove Proposition 1.4.6. As mentioned in the Introduction, the proof resembles the one for the Schur criterion stating that an operator is bounded on $L^2$ if its Schwartz kernel is uniformly $L^1$ in each variable separately (see e.g. [Sai91, Lemma 3.7]). Hence it is necessary to understand well the properties of the kernels of $\overline{A}_{\chi_0,\eta,\sigma}$ and $\widehat{A}_{\chi_0,\eta,\sigma}$. The fine behavior of these kernels is perhaps best described in terms of the scattering blow-up of the product $\overline{X}^2$. We begin by describing blow-ups in general and subsequently the scattering blow-up, in Section 1.5.1. We then analyze the kernels on it in Section 1.5.2.

### 1.5.1 Blown-Up Spaces

In this section we describe the scattering blow-up of the product $\overline{X}^2$, where $\overline{X}$ is a compact manifold with boundary. A reference for blow-ups in general is [Mel]; specifically for the scattering blow-up see [Mel94]. Let $\overline{X}$ be an $n$-dimensional compact manifold with corners and $Z$ a p-submanifold of codimension at least 2 (see [Mel, Definition 1.7.4]); this means that $Z$ is a submanifold of $\overline{X}$ with the property that for each $p \in Z$ there exists a coordinate chart $(U, \varphi)$ for $\overline{X}$ centered at $p$ and integers $0 \leq k \leq n$, $0 \leq r \leq k$, $0 \leq s \leq n-k$ with $r + s \geq 2$ such that

$$\varphi(Z \cap U) = \{(x, y) \in \overline{\mathbb{R}}^k_+ \times \mathbb{R}^{n-k} : x_1 = \cdots = x_r = y_1 = \cdots = y_s = 0\}, \tag{1.5.1}$$

where if $r = 0$ or $s = 0$ we mean that none of the $x_i$ or $y_j$ respectively vanish identically on $\varphi(Z \cap U)$. In the case when $r = 0$ in (1.5.1) (in which case $Z$ is called an *interior p-submanifold*), blowing up $Z$ essentially amounts to introducing polar coordinates in terms of $(y_1, \ldots, y_s)$. Formally, one defines the spherical normal bundle of $Z$, $SN(Z) \xrightarrow{\pi} Z$, with fiber at $p \in Z$ given by $SN_p(Z) := \left((T_p\overline{X}/T_pZ)\backslash\{0\}\right)/\mathbb{R}^+$. Then, it can be shown ([Mel], Sections 5.1-5.3) that the *blown up* space $\left[\overline{X}; Z\right] := SN(Z) \amalg (\overline{X}\backslash Z)$ admits a smooth structure as a manifold with corners such the blow down map $\beta : \left[\overline{X}; Z\right] \to \overline{X}$, given by $\beta\big|_{\overline{X}\backslash Z} = Id_{\overline{X}\backslash Z}$ and $\beta\big|_{SN(Z)} = \pi$, becomes smooth with rank $\dim(Z) + 1$ on $SN(Z)$. In the case when $Z$ is a *boundary p-submanifold*, meaning that $r > 0$ in (1.5.1), the spherical normal bundle is replaced by its inward pointing part, with fiber $S^+N_p(Z) = \left((T_p^+\overline{X}/T_pZ)\backslash\{0\}\right)/\mathbb{R}^+$ and the rest of the discussion follows in the same way as for interior p-submanifolds. If $P$ is a p-submanifold of $\overline{X}$ that intersects $Z$, with the property that $\overline{(P\backslash Z)} = P$, and $\beta$ is a blow down map, then the *lift* of $P$ is defined as $\beta^*(P) = \overline{\beta^{-1}(P\backslash Z)}$.

Now let $\overline{X}$ be a smooth compact manifold with boundary; this implies that $\overline{X}^2$ is a smooth manifold with corners. We will define a number of spaces originating from $\overline{X}^2$ after blowing up successively certain p-submanifolds. Following [Mel94], define the $b$-space $\overline{X}^2_b := \left[\overline{X}^2; (\partial\overline{X})^2\right]$ with blow down map $\beta_1$. We denote by $\mathrm{ff}_b$ the front face of this blow up. If $\Delta_b := \overline{\beta_1^{-1}(\Delta^\circ)}$ (where $\Delta \subset \overline{X}^2$ is the diagonal, which is not a p-submanifold), we let the scattering product be $X^2_{sc} := \left[\overline{X}^2_b; \partial(\Delta_b)\right]$ and the blow down map be $\beta_2 : \overline{X}^2_{sc} \to \overline{X}^2_b$. Set $\beta_{sc} = \beta_1 \circ \beta_2$ and let $\mathrm{ff}_{sc} \subset \overline{X}^2_{sc}$ be the front face associated to $\beta_2$. We finally introduce a third blown up space obtained from $\overline{X}^2_{sc}$ by blowing up the scattering diagonal $\Delta_s := \beta_2^*(\Delta_b)$. We denote the new space by $\overline{X}^2_{\Delta_s}$ and the corresponding blow down map by $\beta_3$; let $\beta_{\Delta_s} := \beta_{sc} \circ \beta_3$. This space is pictured in Figure 1.2. By a result on commutativity of blow-ups (see [Mel, Section 5.8]), $\overline{X}^2_{\Delta_s}$ is diffeomorphic to $\left[[\overline{X}^2_b; \Delta_b], \widetilde{\beta}_2^*(\partial\Delta_b)\right]$, where $\widetilde{\beta}_2 : [\overline{X}^2_b; \Delta_b] \to \overline{X}^2_b$ is the blow down map. We name the various faces of $X^2_{\Delta_s}$ as follows:

$$\mathcal{G}_{10} := \beta_3^*\left(\beta_2^*\left(\beta_1^*\left(\partial\overline{X}\times\overline{X}\right)\right)\right) \quad \mathcal{G}_{01} := \beta_3^*\left(\beta_2^*\left(\beta_1^*\left(\overline{X}\times\partial\overline{X}\right)\right)\right)$$
$$\mathcal{G}_{11} := \beta_3^*\left(\beta_2^*\left(\mathrm{ff}_b\right)\right) \qquad\qquad \mathcal{G}_2 := \beta_3^*(\mathrm{ff}_{sc});$$

finally let $\mathcal{G}_3$ be the front face associated with $\beta_3$. Introducing some more notation, let $p \in \partial\overline{X}$ and $U$ be a neighborhood of $p$ in $\overline{X}$ or the closure of one. Then we let by definition $O^2_b$, $O^2_{sc}$ and $O^2_{\Delta_s}$ mean $\beta_1^{-1}(O^2)$, $\beta_{sc}^{-1}(O^2)$ and $\beta_{\Delta_s}^{-1}(O^2)$ respectively.

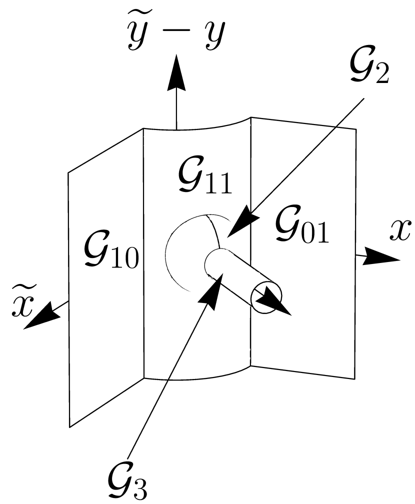


Figure 1.2: The scattering product space $\overline{X}^2_{\Delta_s}$.

The blown up spaces mentioned above are conveniently studied using projective coordinates. Let $(x, y)$ and $(\widetilde{x}, \widetilde{y})$ be two copies of the same coordinate system in a neighborhood $O$ of a point $p \in \partial\overline{X}$, so that $(x, y, \widetilde{x}, \widetilde{y})$ is a coordinate system for $O^2 \subset \overline{X}^2$, and change the dimension convention to $\dim\left(\overline{X}\right) = n + 1$. Here and for the rest of this section $x$ (and thus also $\widetilde{x}$) is a boundary defining function for $\partial X$. Then the projective coordinate systems $(s_1 = \widetilde{x}/x, x, y, \widetilde{y})$ and $(s_2 = x/\widetilde{x}, \widetilde{x}, y, \widetilde{y})$ are valid in a neighborhood of $\mathcal{G}_{01}$ and $\mathcal{G}_{10}$ respectively and the coordinate functions are smooth away from $\mathcal{G}_{10}$ and $\mathcal{G}_{01}$ respectively (though they do not form coordinate systems near $\mathcal{G}_2$ and $\mathcal{G}_3$ in $\overline{X}^2_{\Delta_s}$). In terms of the former coordinate system, $s_1$ is a defining function for $\mathcal{G}_{01}$ and $x$ a defining function for $\mathcal{G}_{11}$, whereas in terms of the latter $s_2$ is a defining function for $\mathcal{G}_{10}$ and $\widetilde{x}$ is one for $\mathcal{G}_{11}$. On the other hand, either by checking directly or by using the commutativity of the blow-up mentioned before, one sees that a valid coordinate system in a neighborhood of any point near $\mathcal{G}_{11} \cap \mathcal{G}_2$ can be obtained by appropriately choosing $n$ of the $\theta^j$ below:

$$\left(\tau = \sqrt{(s_1-1)^2 + |\widetilde{y}-y|^2}, \theta = \frac{(s_1-1, \widetilde{y}-y)}{\tau}, \sigma = \frac{x}{\tau}, y\right). \tag{1.5.2}$$

In (1.5.2) $|\cdot|$ denotes the Euclidean norm. For instance, letting $\varepsilon > 0$ be small and $U_J^{\pm} = \{(\hat{X}, \hat{Y}) = (\theta^0, \ldots, \theta^n) \in \mathbb{S}^n : \pm\theta^J > \varepsilon\}$, $J = 0, \ldots, n$ we can cover $\mathbb{S}^n$ by the $U_J^{\pm}$ and use $\theta^j$, $j \neq J$ as smooth coordinates on $U^J_{\pm}$ for each choice of $\pm$. Now note that $(X = (s_1-1)/x, Y = (\widetilde{y}-y)/x, x, y)$ are valid coordinates on $(O^2_{\Delta_s})^\circ$, and smooth up to $\mathcal{G}_3$ and $\mathcal{G}_2^\circ$. Thus one obtains a diffeomorphism $\mathcal{T}$ from the interior of $O^2_{\Delta_s}$ onto an open subset of $\overline{\mathbb{R}^{n+1}_+} \times [0,\infty) \times \mathbb{S}^n$ that extends smoothly up to $\mathcal{G}_3$ and $\mathcal{G}_2^\circ$ by setting

$$\left(x, y, R = \sqrt{X^2 + |Y|^2}, \theta = (\hat{X}, \hat{Y}) = \frac{(X,Y)}{R}\right) \in \overline{\mathbb{R}^{n+1}_+} \times [0,\infty) \times \mathbb{S}^n. \tag{1.5.3}$$

Again we can choose coordinates on $\mathbb{S}^n$ to obtain coordinate systems on $(O^2_{\Delta_s})^\circ$, smooth up to $\mathcal{G}_3$ and $\mathcal{G}_2^\circ$. Note that $\theta = (\theta_0, \ldots, \theta_n)$ stands for the same functions in both (1.5.2) and (1.5.3) and that $R$ is a defining function for $\mathcal{G}_3$. Moreover, we note here that that

$$\begin{aligned}
x_{01} =& \frac{1 + xR\hat{X}}{2 + xR\hat{X}} = \frac{s_1}{1+s_1}, \qquad x_{10} = (2 + xR\hat{X})^{-1} = \frac{s_2}{1+s_2} \\
x_{11} =& \frac{(2+xR\hat{X})^2}{1+R} = \frac{\sigma}{1+\sigma}(2+\tau\theta^0)^2, \\
=& \frac{x(1+s_1)^2}{x + \sqrt{(s_1-1)^2 + |\widetilde{y}-y|^2}} = \frac{\widetilde{x}(s_2+1)^2}{s_2^2\widetilde{x} + \sqrt{(1-s_2)^2 + s_2^2|\widetilde{y}-y|^2}}
\end{aligned} \tag{1.5.4}$$

are smooth defining functions for $\mathcal{G}_{01}$, $\mathcal{G}_{10}$ and $\mathcal{G}_{11}$ respectively, each non-vanishing and smooth up to all other faces.

Via the diffeomorphism $\mathcal{T}$ the expression $|dx\, dy\, dR\, d\omega|$ (where $d\omega$ is the volume form on $\mathbb{S}^n$ induced by the round metric) pulls back to a smooth global section of the density bundle on $(O^2_{\Delta_s})^\circ$, which is smooth and non-vanishing up to $\mathcal{G}_3$ and $\mathcal{G}_2^\circ$, but not up to the other boundary faces. We have the following:

**Lemma 1.5.1.** *Via the diffeomorphism $\mathcal{T}$ defined by* (1.5.3)*, the expression*

$$(R+1)^{-1}(2+xR\hat{X})^n|dx\,dy\,dR\,d\omega| \tag{1.5.5}$$

*pulls back to a smooth non-vanishing section of the smooth density bundle on $O^2_{\Delta_s}$, up to all boundary faces.*

*Proof.* The claim can be checked via a straightforward computation in local coordinates smooth up to the various boundary faces in different parts of $O^2_{\Delta_s}$. Near $\mathcal{G}_3$ and $\mathcal{G}_2^\circ$ $(R+1)^{-1}(2+xR\hat{X})^n$ is smooth and non-vanishing, so there is nothing to show there (note that $2+xR\hat{X} = s_1+1$ and $s_1>0$ in $(O^2_{\Delta_s})^\circ$). Then we compute

$$\begin{aligned}\frac{(2+xR\hat{X})^n}{1+R}|dx\,dy\,dR\,d\omega| =& \frac{(2+\tau\theta^0)^n}{1+\sigma}|d\tau\,dy\,d\sigma\,d\omega| && (1.5.6)\\ =& \frac{(s_1+1)^n\big((s_1-1)^2+|\widetilde{y}-y|^2\big)^{-\frac{n}{2}}}{x+\sqrt{(s_1-1)^2+|\widetilde{y}-y|^2}}|dx\,dy\,ds_1\,d\widetilde{y}| && (1.5.7)\\ =& \frac{(1+s_2)^n\big((1-s_2)^2+s_2^2|\widetilde{y}-y|^2\big)^{-\frac{n}{2}}}{s_2^2\widetilde{x}+\sqrt{(1-s_2)^2+s_2^2|\widetilde{y}-y|^2}}|ds_2\,dy\,d\widetilde{x}\,d\widetilde{y}|. && (1.5.8)\end{aligned}$$

Then (1.5.6) shows the claim near $\mathcal{G}_{11}\cap\mathcal{G}_2$ and away from the other intersections of boundary faces, (1.5.7) near $\mathcal{G}_{01}$ and (1.5.8) near $\mathcal{G}_{10}$. □

We now record the form that $\beta^*_{\Delta_s}\widetilde{W}$, the lift of $\widetilde{W}$, takes in terms of (1.5.3) whenever $\widetilde{W}\in\mathcal{V}_{sc}(\overline{X})$ is identified with a vector field on $\overline{X}^2$ acting on the left factor. Here the lift is well defined since $\beta_{\Delta_s}:(\overline{X}^2_{\Delta_s})^\circ\to X^2\setminus\Delta$ is a diffeomorphism. We work in a neighborhood $O^2$ of a point $(p,p)\in\partial\Delta$ where we have coordinates $(x,y,\widetilde{x},\widetilde{y})$, as before. Then $\widetilde{W}$ is spanned over $C^\infty$ by $x^2\partial_x$, $x\partial_{y^\alpha}$. Those lift via $\beta_1$ to the vector fields $-xs_1\partial_{s_1}+x^2\partial_x$, $x\partial_{y^\alpha}$ respectively, in coordinates $(x, s_1=\widetilde{x}/x, y, \widetilde{y})$. Now we lift those using $\beta_2$ and find that in terms of coordinates $(x,y,X,Y)$ they are given respectively by $(-1-2xX)\partial_X - xY\cdot\partial_Y + x^2\partial_x$ and $-\partial_Y + x\partial_{y^\alpha}$. Now blowing up $\Delta_s$ corresponds to using polar coordinates about $(X,Y)=0$.

Again consider the sets $U_J^{\pm}$, $J=0,\dots,n$, defined before, for some fixed small $\varepsilon$: on $U_J^{\pm}$ the functions $\theta^j$, $j\neq J$, form a smooth coordinate system. Then check that for each $J$ and choice of $\pm$ there exist smooth functions $a^j_{J,\pm}$ and $b^j_{J,\pm}$ on $U^{\pm}_{J,\pm}$ such that the vector fields $\hat{X}\partial_R + R^{-1}\sum_{j\neq J}^n a^j_{J,\pm}(\theta)\partial_{\theta^j}$ and $\hat{Y}^\alpha\partial_R + R^{-1}\sum_{j\neq J} b^j_{J,\pm}(\theta)\partial_{\theta^j}$ are $\beta_3$-related to $\partial_X$ and $\partial_{Y^\alpha}$ respectively. Thus if $\widetilde{W}\in\{x^2\partial_x, x\partial_y\}$ then in the set $\{(x,y,R,\theta)\in O\times[0,\infty)\times U_J^{\pm}\}$ we have $\beta^*_{\Delta_s}\widetilde{W}=\sum_j c^j_{J,\pm}(x,y,R,\theta)\,W_j$, where $W_j$ belong to either of the two sets $\{x^2\partial_x, x\partial_y,\partial_R\}$ or $\{R^{-1}\partial_{\theta^j}, j\neq J\}$ and $c^j_{J,\pm}$ are smooth and grow at most polynomially fast as $R\to\infty$ (though note that $\beta^*_{\Delta_s}\widetilde{W}$ is smooth on $\overline{X}^2_{\Delta_s}\setminus\mathcal{G}_3$).

### *1.5.2 Analysis on blow-ups*

In this section we identify the form of the Schwartz kernels of the operators $A_{\chi,\eta,\sigma}$ defined in Section 1.4 (Lemma 1.5.2) and prove two technical lemmas regarding their regularity and dependence on the parameter $\eta$ when lifted to the scattering stretched product space (Lemmas 1.5.3 and 1.5.4). We then use those to analyze the kernel of the difference $E_{\eta,\sigma}$ in Lemma 1.5.6 and finally its properties to prove Proposition 1.4.6.

Recall that the operators $A_{\chi,\eta,\sigma}$ act on functions supported in sets varying with the parameter $\eta$. As in [UV16], it will be convenient to create an auxiliary 1-parameter family of operators which all act on functions defined on the same space. We use the smooth 1-parameter family of maps $\psi_\eta(\cdot)$, defined in Section 1.4, namely the flow of $\operatorname{grad}\hat{x}/|\operatorname{grad}\hat{x}|^2_{g^0}$, to map diffeomorphically $\overline{X}_\eta$ onto $\overline{X}_0$ (locally in $\mathcal{U}$). For $\sigma>0$, $\eta\geq 0$ and $\chi$ as in Section 1.4 define a 1-parameter family of operators by

$$\widetilde{A}_{\chi,\eta,\sigma} := (\psi_{-\eta})^*\circ A_{\chi,\eta,\sigma}\circ(\psi_\eta)^*, \tag{1.5.9}$$

all acting on functions supported in $\overline{X}_0$ near $p$. We use the notation $\widetilde{\overline{A}}_{\chi,\eta,\sigma}$ and $\widetilde{\widehat{A}}_{\chi,\eta,\sigma}$ for the operators corresponding to $\nabla=\overline{\nabla}$ and $\widehat{\nabla}$. Similarly, for $\chi_0$ determined by Proposition 1.4.3 let

$$\widetilde{E}_{\eta,\sigma} := \widetilde{\overline{A}}_{\chi_0,\eta,\sigma} - \widetilde{\widehat{A}}_{\chi_0,\eta,\sigma}. \tag{1.5.10}$$

Proposition 1.4.6 immediately reduces to showing the following:

*Let $\sigma > 0$. Provided $O$ is a sufficiently small neighborhood $p$ in $\overline{X}_0$, for every $\delta > 0$ there exits $\eta_0 > 0$ with the property that if $0 \le \eta < \eta_0$ one has $\widetilde{U}_\eta := \psi_\eta(U_\eta) \subset\subset O$ and*

$$\|\widetilde{E}_{\eta,\sigma} u\|_{H^{1,0}_{sc}(O)} \le \delta \|u\|_{L^2(\widetilde{U}_\eta)}, \tag{1.5.11}$$

*for all $u \in L^2(\widetilde{U}_\eta)$ extended by 0 outside $\widetilde{U}_\eta$.*

We will now use the product decomposition $[0, \delta_0)_{x_0} \times Y_p$ introduced in Section 1.4 to analyze the Schwartz kernels of (1.5.9) and (1.5.10) on $\overline{X}_0^2$. Henceforth we will write $g$ for the metric $g^0$ which was chosen in Section 1.3 and was used to define $A$ in (1.4.3), and $S\widetilde{M}$ for its unit sphere bundle. No confusion will arise with the AH metric $g$, as it will not appear again.

**Lemma 1.5.2.** *Suppose $\nabla$ is a connection on $T\widetilde{M}$ whose exponential map $\exp : T\widetilde{M} \to \widetilde{M}$ is of class $C^2$ and for which $\partial \overline{M}_e$ is strictly convex. Also let $\chi \in C_c^\infty(\mathbb{R})$ be even with $\chi(0) = 1$, $\chi \ge 0$, and let $\sigma > 0$. Let $\kappa_{\widetilde{A}_{\chi,\eta,\sigma}}$ denote the Schwartz kernel of $A_{\chi,\eta,\sigma}$, viewed as a section of the smooth density bundle on $\overline{X}_0^2$. Then*

$$\kappa_{\widetilde{A}_{\chi,\eta,\sigma}} = \sqrt{g(z-\overline{\eta})}\, x^{-2} e^{-\sigma(1/x - 1/\widetilde{x})} 2\chi\left(P(z,\widetilde{z},\eta)\right) \frac{|\det(d_{\widetilde{z}} \exp^{-1}_{z-\overline{\eta}})(\widetilde{z}-\overline{\eta})|}{|\exp^{-1}_{z-\overline{\eta}}(\widetilde{z}-\overline{\eta})|_g^n} |dz\, d\widetilde{z}|,$$
$$\text{where } P(z,\widetilde{z},\eta) := \frac{dx_0\big(\exp^{-1}_{z-\overline{\eta}}(\widetilde{z}-\overline{\eta})\big)}{x\big|dy\big(\exp^{-1}_{z-\overline{\eta}}(\widetilde{z}-\overline{\eta})\big)\big|_g} \text{ and } \overline{\eta} = (\eta, 0), \tag{1.5.12}$$

*for $(z, \widetilde{z})$ in a sufficiently small neighborhood of $(p,p)$ written as $z = (x,y)$, $\widetilde{z} = (\widetilde{x}, \widetilde{y})$ in terms of the product decomposition $[0,\delta_0)_{x_0} \times Y_p$ and with $y$, $\widetilde{y}$ identical copies of a coordinate system on $Y_p$ centered at $p$. Here and in what follows $\sqrt{g(z-\overline{\eta})} = \sqrt{\det g(z-\overline{\eta})}$ with the determinant computed in terms of the coordinates $(x,y)$.*

*Proof.* First examine the kernel of $A_{\chi,\eta,\sigma}$ on $\overline{X}_\eta^2$, for fixed $\eta \ge 0$ small. Let $f$ be smooth and supported in a small neighborhood in $\overline{X}_\eta$ of a point in $U_\eta \subset \overline{X}_\eta$. We write $z' = (x', y)$, $\widetilde{z}' = (\widetilde{x}', \widetilde{y})$ in terms of the product decomposition $[0, \delta_0)_{x_\eta} \times Y_p$ on $\overline{X}_\eta^2$ with $y$, $\widetilde{y}$ coordinates on $Y_p$ and also $v' = \lambda' \partial_{x'} + \omega'$ for vectors in $T_{z'}\overline{X}_\eta$. Writing $d\lambda_g$ for the measure induced by $g$ on each fiber of $T\widetilde{M}$, compute

$$A_{\chi,\eta,\sigma} f(z') = x'^{-2} e^{-\sigma/x'} \int_{S_{z'}\overline{X}_\eta} \chi\left(\frac{\lambda'}{x'|\omega'|_g}\right) \int_{-\infty}^{\infty} (e^{\sigma/x_\eta} f)\big|_{\widetilde{z}' = \exp_{z'}(tv')} dt\, d\mu_g$$

$$
\begin{aligned}
&= x'^{-2} e^{-\sigma/x'} \int_{S_{z'}\overline{X}_\eta} 2\chi\left(\frac{\lambda'}{x'|\omega'|_g}\right) \int_0^\infty (e^{\sigma/x_\eta} f)\big|_{\widetilde{z}'=\exp_{z'}(tv')} dt\, d\mu_g \\
&= x'^{-2} e^{-\sigma/x'} \int_{T_{z'}\overline{X}_\eta} 2\chi\left(\frac{\lambda'}{x'|\omega'|_g}\right) (e^{\sigma/x_\eta} f)\big|_{\widetilde{z}'=\exp_{z'}(tv')} \frac{d\lambda_g}{|v'|_g^n} \\
&= x'^{-2} e^{-\sigma/x'} \int_{\overline{X}_\eta} 2\chi\left(\frac{dx_\eta\big(\exp_{z'}^{-1}(\widetilde{z}')\big)}{x'|dy\big(\exp_{z'}^{-1}(\widetilde{z}')\big)|_g}\right) \frac{e^{\sigma/\widetilde{x}'} f(\widetilde{z}')}{|\exp_{z'}^{-1}(\widetilde{z}')|_g^n} (\exp_{z'})_*(d\lambda_g).
\end{aligned}
$$

Recall that by Lemma 1.4.2 the two integrals with respect to $t$ above are in fact over finite intervals $(-\delta_1, \delta_1)$ and $[0, \delta_1)$, respectively. Moreover, $d\lambda_g(v') = \sqrt{g(z')}|dv'|$ in terms of induced fiber coordinates. Finally conjugation by $\psi_\eta$ in (1.5.9) corresponds to replacing $(z', \widetilde{z}')$ by $(z - \overline{\eta}, \widetilde{z} - \overline{\eta})$ in the Schwartz kernel of $A_{\chi,\eta,\sigma}$, where $z$, $\widetilde{z}$ are expressed in terms of the product decomposition $[0, \delta_0)_{x_0} \times Y_p$ on $\overline{X}_0$. Noting that $dx_\eta = dx_0$ completes the proof. $\square$

In the next two lemmas we use (1.5.12) to analyze the Schwartz kernel of $\widetilde{A}_{\chi,\eta,\sigma}$ on $\left(\overline{X}_0\right)^2_{\Delta_s}$ near $\beta_{\Delta_s}^{-1}(p,p)$. Since the proof of Proposition 1.4.6 has been reduced to showing (1.5.11), from now on the entire analysis will be on $\overline{X}_0$. We will thus drop the subscript and write $\overline{X}$ to mean $\overline{X}_0$. We will use the product decomposition $[0, \delta_0)_{x_0} \times Y_p$ near $\partial\overline{X}$ introduced in Section 1.4 in each factor of $\overline{X}$. Again we choose coordinates $y^\alpha$ on $Y_p$ centered at $p$, $\alpha = 1, \ldots, n$, and write $z = (z^0, z^\alpha) = (x, y^\alpha)$ and $\widetilde{z} = (\widetilde{z}^0, \widetilde{z}^\alpha) = (\widetilde{x}, \widetilde{y}^\alpha)$ for points in the left and right factor of $\overline{X}$ respectively. We will also be using the notations $\mathcal{G}_*$ introduced in Section 1.5.1 to describe the various boundary faces of $\overline{X}^2_{\Delta_s}$. In what follows, whenever we say that a function $f$ vanishes to infinite order at a boundary face of a manifold with corners, we will mean that if $x$ is a defining function of this boundary face and $N_0 \in \mathbb{N}$ then $x^{-N_0} f \in L^\infty$ (thus this is purely a statement regarding the growth of $f$ without any mention of the behavior of its derivatives near $x = 0$).

**Lemma 1.5.3.** *Let the hypotheses of Lemma 1.5.2 hold and let $\nu$ be a section of the smooth density bundle on $\overline{X}^2_{\Delta_s}$. For a sufficiently small neighborhood $O$ of $p$ in $\overline{X}$ there exists $\eta_0$ depending on $O$, $\nabla$ and $\chi$ such that for the Schwartz kernel $\kappa_{\widetilde{A}_{\chi,\eta,\sigma}}$ of $\widetilde{A}_{\chi,\eta,\sigma}$ computed in*

(1.5.12) *one has*

$$\beta^*_{\Delta_s}(\kappa_{\widetilde{A}_{\chi,\eta,\sigma}}) = K_\nabla(\cdot,\eta)\cdot\nu, \quad K_\nabla(\cdot,\cdot)\in C^0\big(O^2_{\Delta_s}\times[0,\eta_0)\big).$$

*Moreover,* $K_\nabla \in C^1\big(((O^2_{\Delta_s})^\circ \cup \mathcal{G}_3^\circ\big)\times[0,\eta_0)\big)$*, it vanishes to infinite order on faces* $\mathcal{G}_{10}$*,* $\mathcal{G}_{11}$ *and* $\mathcal{G}_{01}$ *and its restriction to* $\mathcal{G}_3$ *is independent of* $\nabla$*.*

*Proof.* We will lift (1.5.12) to $\overline{X}^2_{\Delta_s}$ and analyze its regularity. We always assume that we are working in a small enough neighborhood $O^2$ and with small enough $\eta_0$ that the coordinates $(x,y,\widetilde{x},\widetilde{y})$ are valid, $\exp^{-1}_{z-\overline{\eta}}(\widetilde{z}-\overline{\eta})$ is a $C^2$ diffeomorphism onto its image for $(z,\widetilde{z})\in O^2$ and $0\le\eta<\eta_0$, and the conclusion of Lemma 1.4.2 holds.

Before we lift (1.5.12) to the stretched product we analyze its various factors on $\overline{X}^2$. Use Taylor's Theorem for the function $t\mapsto \exp^{-1}_{z-\overline{\eta}}(z-\overline{\eta}+t(\widetilde{z}-z))$ and write two different expressions for $\exp^{-1}_{z-\overline{\eta}}(\widetilde{z}-\overline{\eta})$ in terms of coordinates $z=(x,y^\alpha)=(z^0,z^\alpha)$, $\alpha=1,\dots,n$:

$$\begin{aligned} dz^k(\exp^{-1}_{z-\overline{\eta}}(\widetilde{z}-\overline{\eta})) =& p^k_j(z,\widetilde{z},\eta)(\widetilde{z}-z)^j && (1.5.13)\\ =&(\widetilde{z}-z)^k + p^k_{ij}(z,\widetilde{z},\eta)(\widetilde{z}-z)^i(\widetilde{z}-z)^j, \quad \text{where} && (1.5.14)\end{aligned}$$

$$p^k_j(z,\widetilde{z},\eta) := \int_0^1 \partial_{\widetilde{z}^j}\left(dz^k\exp^{-1}_{z-\overline{\eta}}\right)\big|_{z-\overline{\eta}+\tau(\widetilde{z}-z)}d\tau \in C^1\big(O^2\times[0,\eta_0)\big),$$

$$p^k_{ij}(z,\widetilde{z},\eta) := \int_0^1 (1-\tau)\partial_{\widetilde{z}^i\widetilde{z}^j}\left(dz^k\exp^{-1}_{z-\overline{\eta}}\right)\big|_{z-\overline{\eta}+\tau(\widetilde{z}-z)}d\tau \in C^0\big(O^2\times[0,\eta_0)\big),$$

with

$$p^k_j(z,z,\eta)=\delta^k_j \quad\text{and}\quad p^k_{ij}(z,z,\eta)=\tfrac{1}{2}\Gamma^k_{ij}(z-\overline{\eta}).$$

Here $\Gamma^k_{ij}$ denote the connection coefficients of $\nabla$ in coordinates $(x,y)$. Now (1.5.13) and (1.5.14) can be used to show regularity of the factors of 1.5.12. We have

$$|\det(d_{\widetilde{z}}\exp^{-1}_{z-\overline{\eta}})(\widetilde{z}-\overline{\eta})|\in C^1\big(O^2\times[0,\eta_0)\big), \quad |\det(d_{\widetilde{z}}\exp^{-1}_{z-\overline{\eta}})(z-\overline{\eta})|=1. \tag{1.5.15}$$

Using (1.5.13) and the smoothness of the metric $g$,

$$|\exp^{-1}_{z-\overline{\eta}}(\widetilde{z}-\overline{\eta})|^2_g = G_{ij}(z,\widetilde{z},\eta)(\widetilde{z}-z)^i(\widetilde{z}-z)^j, \text{ where } G_{ij}\in C^1\big(O^2\times[0,\eta_0)\big),$$

$$G_{ij}(z,z,\eta) = g_{ij}(z-\overline{\eta}),\ G_{ij} \text{ positive definite in } O^2 \times [0,\eta_0). \tag{1.5.16}$$

To analyze $P$ from (1.5.12) write, using (1.5.13) and (1.5.14),

$$\begin{aligned} P(z,\widetilde{z},\eta) =& \frac{p^0_j(z,\widetilde{z},\eta)(\widetilde{z}-z)^j}{x\big(q_{ij}(z,\widetilde{z},\eta)(\widetilde{z}-z)^i(\widetilde{z}-z)^j\big)^{1/2}}, \\ =& \frac{\widetilde{x}-x+p^0_{ij}(z,\widetilde{z},\eta)(\widetilde{z}-z)^i(\widetilde{z}-z)^j}{x\big(|\widetilde{y}-y|^2_g + q_{ijk}(z,\widetilde{z},\eta)(\widetilde{z}-z)^i(\widetilde{z}-z)^j(\widetilde{z}-z)^k\big)^{1/2}}, \end{aligned}$$

where $q_{ij} = g_{\alpha\beta}p^\alpha_i p^\beta_j \in C^1\big(O^2\times[0,\eta_0)\big)$, $q_{ijk} \in C^0\big(O^2\times[0,\eta_0)\big)$.

We now examine the lifts of the various factors of the kernel to $O^2_{\Delta_s}$. As explained in Section 1.5.1, near any point in $(O^2_{\Delta_s})^\circ$, $2n+2$ of the functions $\big(x, y, R = \sqrt{X^2+|Y|^2}, \theta = (\hat{X},\hat{Y}) = (X,Y)/R\big)$, where $X = (\widetilde{x}-x)/x^2,\ Y = (\widetilde{y}-y)/x$, form a smooth coordinate system; moreover, the functions $(x,y,R,\theta)$ are smooth up to $\mathcal{G}_2^\circ$ and $\mathcal{G}_3$, and $x$ is a defining function for $\mathcal{G}_2$ and $R$ is a defining function for $\mathcal{G}_3$. First note that since $\beta_{\Delta_s}$ is smooth, (1.5.15) implies that

$$\beta^*_{\Delta_s}(|\det(d_{\widetilde{z}}\exp^{-1}_{z-\overline{\eta}})(\widetilde{z}-\overline{\eta})|) \in C^1\big(O^2_{\Delta_s}\times[0,\eta_0)\big)$$

and it is identically 1 at $\mathcal{G}_2$ and $\mathcal{G}_3$. Writing $\hat{Z} = (x\hat{X},\hat{Y})$ so that $\widetilde{z}-z = xR\hat{Z}$, (1.5.16) yields

$$\begin{aligned} \beta^*_{\Delta_s}|\exp^{-1}_{z-\overline{\eta}}(\widetilde{z}-\overline{\eta})|^{-n}_g =& x^{-n}R^{-n}\left(G_{ij}(z,z+xR\hat{Z},\eta)\hat{Z}^i\hat{Z}^j\right)^{-n/2} \\ =& x^{-n}R^{-n}\left(G_{\alpha\beta}\hat{Y}^\alpha\hat{Y}^\beta + 2xG_{0\beta}\hat{X}\hat{Y}^\beta + x^2G_{00}\hat{X}^2\right)^{-n/2} \end{aligned}$$

and we also have

$$\beta^*_{\Delta_s}|dx\,dy\,d\widetilde{x}\,d\widetilde{y}| = x^{n+2}R^n|dx\,dy\,dR\,d\omega|.$$

We now pull back $\chi(P)$, writing it in two ways using (1.5.13) and (1.5.14):

$$\beta^*_{\Delta_s}(\chi(P)) = \chi\left(\frac{p^0_j(z,z+xR\hat{Z},\eta)\hat{Z}^j}{x\left(q_{ij}(z,z+xR\hat{Z},\eta)\hat{Z}^i\hat{Z}^j\right)^{1/2}}\right) \tag{1.5.17}$$

$$=\chi\left(\frac{\hat{X}+R\,p^0_{\alpha\beta}\hat{Y}^\alpha\hat{Y}^\beta+xR\left(\,2p^0_{0\beta}\hat{X}\hat{Y}^\beta+xp^0_{00}\hat{X}^2\right)}{\left(|\hat{Y}|^2_g+xR\,q_{ijk}\hat{Z}^i\hat{Z}^j\hat{Z}^k\right)^{1/2}}\right), \tag{1.5.18}$$

where in (1.5.18) the $p^k_{ij}$ and $q_{ijk}$ are all evaluated at $(z, z+xR\hat{Z}, \eta)$. Some caution is required when the denominator of the arguments approaches 0. At any point in $(O^2_{\Delta_s})^\circ \times [0, \eta_0)$ the expression $\beta^*_{\Delta_s}(\chi(P))$ is $C^2$, since any such point projects via $\beta_{\Delta_s}$ to a pair of points away from the diagonal. This implies that if the denominator of $P$ vanishes the numerator does not, and hence $\chi(P) = 0$ there. Now suppose we are given $q' = (x', y', R', \theta', \eta') \in \mathcal{G}^\circ_2 \cup \mathcal{G}_3 \times [0, \eta_0)$, so either $x' = 0$ or $R' = 0$. Since $|\theta| = |(\hat{X}, \hat{Y})| = 1$, either $\hat{X}$ or $\hat{Y}$ are bounded away from 0. If $|\hat{Y}| \leq \varepsilon$ for some $\varepsilon > 0$, the numerator of $P$ is bounded below in absolute value by $\sqrt{1-\varepsilon^2} - CR(\varepsilon + x)$, therefore the $\varepsilon$ can be assumed to be small enough that in a neighborhood of $q'$ the numerator is bounded below. This again guarantees that $\chi(P)$ is continuous at $q'$ in this case by the compact support of $\chi$. On the other hand, if $|\hat{Y}| \geq \varepsilon$ then in a neighborhood of $q'$ the denominator is bounded away from 0. We conclude that $\beta^*_{\Delta_s}(\chi(P))$ extends continuously to $(O^2_{\Delta_s})^\circ \cup \mathcal{G}_3 \cup \mathcal{G}^\circ_2 \times [0, \eta_0)$ and, in fact, it is in $C^1\big((O^2_{\Delta_s})^\circ \cup \mathcal{G}^\circ_3 \times [0, \eta_0)\big)$ by (1.5.17). A similar analysis applies to show that $R^n x^n \beta^*_{\Delta_s}(|\exp^{-1}_{z-\overline{\eta}}(\widetilde{z}-\overline{\eta})|^{-n}_g) \in C^1\big((O^2_{\Delta_s})^\circ \cup \mathcal{G}^\circ_2 \cup \mathcal{G}_3 \times [0, \eta_0)\big)$ in the support of $\beta^*_{\Delta_s}(\chi(P))$.

Finally, write $\beta^*_{\Delta_s}(x^{-2}e^{-\sigma/x+\sigma/\widetilde{x}}) = x^{-2}e^{-\sigma\frac{R\hat{X}}{1+xR\hat{X}}}$ and combine the lifts of the factors in (1.5.12) together with Lemma 1.5.1 to find

$$\begin{aligned}
&\beta^*(\kappa_{\widetilde{A}^\nabla_{\chi,\eta,\sigma}}) = K_\nabla \cdot \nu \\
&= 2\sqrt{g(z-\overline{\eta})}e^{-\frac{\sigma R\hat{X}}{1+xR\hat{X}}}\chi\big(P(z, z+xR\hat{Z}, \eta)\big)\frac{\left|\det(d_{\widetilde{z}}\exp^{-1}_{z-\overline{\eta}})(z-\overline{\eta}+xR\hat{Z})\right|}{\big(G_{ij}(z, z+xR\hat{Z}, \eta)\hat{Z}^i\hat{Z}^j\big)^{n/2}}\frac{(R+1)}{(2+xR\hat{X})^n}\nu,
\end{aligned} \tag{1.5.19}$$

where $\nu$ is given by (1.5.5) and it is a smooth density on $\overline{X}^2$ independent of $\eta$ and $\nabla$. Together with our analysis of the factors, we conclude that, away from $\mathcal{G}_{10}$, $\mathcal{G}_{11}$ and $\mathcal{G}_{01}$, $K_\nabla \in C^1\big(((O^2_{\Delta_s})^\circ \cup \mathcal{G}^\circ_3) \times [0, \eta_0)\big)$ and continuous up to $\mathcal{G}^\circ_2$. Now Taylor's theorem in terms of $R$ applies for $x > 0$ and we find that for $R > 0$ small and $x > 0$

$$K_\nabla = 2^{-n+1}\chi\left(\frac{\hat{X}}{|\hat{Y}|_g}\right)\sqrt{g(z-\overline{\eta})}|\hat{Z}|_g^{-n} + R\Lambda_\nabla(x,y,R,\theta,\eta), \tag{1.5.20}$$

where $\Lambda_\nabla$ is continuous in all of its arguments, *up to* $x = 0$: observe that by (1.5.17) $\beta^*_{\Delta_s}(\chi(P)) = \chi\left(\frac{a_1(z,xR\hat{Z},\eta,\hat{Z})}{x(a_2(z,xR\hat{Z},\eta,\hat{Z}))^{1/2}}\right)$ with $a_j$ both $C^1$ in their arguments, so by taking an $R$-derivative the chain rule generates a factor of $x$ that cancels the one in the denominator of the argument of $\chi$. Thus $K_\nabla\big|_{\mathcal{G}_3}$ is indeed independent of $\nabla$.

We will now show that $K_\nabla$ vanishes to infinite order on $\mathcal{G}_{10}$, $\mathcal{G}_{11}$ and $\mathcal{G}_{01}$; the arguments here are contained in [UV16] but we repeat them for completeness. By (1.5.4) it suffices to show that $K_\nabla$ decays exponentially as $R \to \infty$, uniformly in $\eta$, and that it vanishes for large negative $X$ (since $x_{01} = 0$ when $xX = -1$). Assume first that we are in the region where $|\hat{Y}| \leq c < 1$ for some constant $c$ uniform in $\eta$, implying that $|\hat{X}|$ is bounded below by a positive constant. If $\hat{X} < 0$ then (1.4.4) guarantees that $\chi(P)$ vanishes identically for sufficiently large $R$ since in its support the variable $X$ is bounded below by a negative constant uniform in $\eta$. On the other hand, when $\hat{X} > 0$, since we are working in a small neighborhood of $p$ and thus $x, \widetilde{x} \leq c_0$ for some small $c_0 > 0$, we have

$$R\hat{X}x = \left(\frac{\widetilde{x}}{x} - 1\right) \Rightarrow R\hat{X}x^2 \leq (c_0 - x) \Rightarrow xR\hat{X} \leq \left(-1 + \sqrt{1 + 4c_0R\hat{X}}\right)/2.$$

Thus $e^{-\sigma\frac{R\hat{X}}{1+xR\hat{X}}} \leq e^{-2\sigma R\hat{X}/\left(1+\sqrt{1+4c_0R\hat{X}}\right)}$, implying that $K_\nabla$ decays exponentially fast, uniformly in $\eta$. Now suppose that $|\hat{Y}| > c/2$. This implies that $p^0_{\alpha\beta}(z, z+xR\hat{X},\eta)\hat{Y}^\alpha\hat{Y}^\beta \leq \widetilde{c} < 0$ if $\eta$ is sufficiently small and $z$ is sufficiently close to $p$, since in $p^0_{\alpha\beta}(z,z,\eta)\hat{Y}^\alpha\hat{Y}^\beta = \Gamma^0_{\alpha\beta}(z-\eta)\hat{Y}^\alpha\hat{Y}^\beta < 0$ (this follows from the strict concavity of $\partial X$ with respect to $\nabla$). Now the triangle inequality yields

$$P(z, z+xR\hat{X},\eta) \geq R^{1/2}\frac{-p^0_{\alpha\beta}\hat{Y}^\alpha\hat{Y}^\beta - |R^{-1}\hat{X} + x(\,2p^0_{\alpha\beta}\hat{X}\hat{Y}^\beta + xp^0_{00}\hat{X}^2)|}{\left(R^{-1}|\hat{Y}|_g^2 + x\,|q_{ijk}\hat{Z}^i\hat{Z}^j\hat{Z}^k|\right)^{1/2}};$$

thus one can choose the $R$ to be large and $x$ to be small enough (by adjusting the size of the neighborhood $O$) uniformly in $\eta$ to guarantee that $\chi(P)$ vanishes identically. This finishes the proof. $\square$

**Lemma 1.5.4.** *Let the hypotheses and notations of Lemma 1.5.3 be in effect. Also let $W$ be the lift to $\overline{X}^2_{\Delta_s}$ of a vector field in $\mathcal{V}_{sc}(\overline{X})$ acting on the left factor of $\overline{X}^2$ and $x_3$ be a defining function for $\mathcal{G}_3$. Then for any sufficiently small neighborhood $O$ of $p$ in $\overline{X}$ there exists $\eta_0 > 0$ such that*

$$x_3 W(K_\nabla) = K_{\nabla,W}(\cdot,\cdot) \in C^0\big(O^2_{\Delta_s} \times [0,\eta_0)\big), \tag{1.5.21}$$

*vanishing to infinite order at $\mathcal{G}_{10}$, $\mathcal{G}_{11}$ and $\mathcal{G}_{01}$. Moreover, whenever $x_3$ induces a product decomposition $\mathcal{G}_3 \times [0,\varepsilon)_{x_3} \times [0,\eta_0)_\eta$ near $\mathcal{G}_3$ (in $O^2_{\Delta_s}$) one has*

$$x_3 W(K_\nabla) = \kappa_W(q,\eta) + x_3 \kappa_{\nabla,W}(q,x_3,\eta), \tag{1.5.22}$$

*where $\kappa_W \in C^0(\mathcal{G}_3 \times [0,\eta_0))$ and independent of $\nabla$ and $\kappa_{\nabla,W} \in C^0(\mathcal{G}_3 \times [0,\varepsilon) \times [0,\eta_0))$.*

**Remark 1.5.5.** *The function $K_\nabla$ is well defined only up to a non-vanishing smooth multiple, since there isn't a completely natural choice of non-vanishing smooth density on $\overline{X}^2_{\Delta_s}$. However it follows from the comments at the end of Section 1.5.1 that $x_3 W$ is smooth on $\overline{X}^2_{\Delta_s}$, hence by Lemma 1.5.3 and* (1.5.20) *it follows that multiplying $K_\nabla$ by a function smooth on $\overline{X}^2_{\Delta_s}$ does not affect the result.*

*Proof.* As discussed in Section 1.5.1 (see (1.5.3)), given a sufficiently small neighborhood $O$ of $p \in \partial\overline{M}_e$ in $\overline{X}$ where global coordinates $(x,y)$ are valid, we obtain an identification of $(O_{\Delta\iota^2_{sc}})^\circ$ with a subset of $\mathbb{R}^{n+1}_+ \times (0,\infty) \times \mathbb{S}^n$. With notation as in Section 1.5.1, we let $\mathcal{U}^\pm_J := \{(x,y,R,\theta,\eta) \in \overline{\mathbb{R}^{n+1}_+} \times [0,\infty) \times U^\pm_J \times [0,\infty)\}$: the union of those sets over $J,\pm$ covers $((O^2_{\Delta_s})^\circ \cup \mathcal{G}^\circ_2 \cup \mathcal{G}_3) \times [0,\eta_0)$ for small $\eta_0 > 0$, and in each of them we have valid coordinates $(x,y,R,\theta^j,\eta)$, $j \neq J$. Now let $W$ be the lift of a scattering vector field acting on the left factor of $\overline{X}^2$. As shown in Section 1.5.1, on $\mathcal{U}^\pm_J$ we have $W = \sum_j c^j_{J,\pm}(x,y,R,\theta)\, W_j$, where $W_j$ belong to either of the two sets $\mathcal{W}_1 = \{x^2\partial_x, x\partial_y, \partial_R\}$ or $\mathcal{W}^J_2 = \{R^{-1}\partial_{\theta^j}, j \neq J\}$ and $c^j_{J,\pm}(x,y,R,\theta)$ are smooth and growing at most polynomially as $R \to \infty$ (i.e. at the faces $\mathcal{G}_{10}$, $\mathcal{G}_{11}$ and $\mathcal{G}_{01}$). Thus it suffices to show (1.5.21) and (1.5.22) for $W_1 \in \mathcal{W}_1$, $W_2 \in \mathcal{W}^J_2$, $J = 0,\ldots,n$.

We will use the expression (1.5.19) we computed for $K_\nabla$ in Lemma 1.5.3. Fix a $J$ and suppose first that $W_1 \in \mathcal{W}_1$: then $W_1$ is smooth for $x > 0$ and we will show continuity of $W_1 K_\nabla$ up to $x = 0$. Recall the notation $\hat{Z} = (x\hat{X}, \hat{Y})$ and observe that

$$\sqrt{g(z-\overline{\eta})}e^{-\sigma\frac{R\hat{X}}{1+xR\hat{X}}}\left|\det(d_{\tilde{z}}\exp^{-1}_{z-\overline{\eta}})(z-\overline{\eta}+xR\hat{Z})\right|\frac{(R+1)}{(2+xR\hat{X})^n} = \Lambda_0(z,R,R\theta,\eta),$$

with $\Lambda_0(z,R,v,\eta) \in C^1\big(O \times [0,\infty) \times \mathbb{R}^{n+1} \times [0,\eta_0)\big)$ for some small neighborhood $O$ of $p$ and small $\eta_0 > 0$. Therefore, $W_1\Lambda_0$ is continuous on the same space. Using (1.5.17), we see that $W_1\big(\chi\big(P(z,z+xR\hat{Z})\big)\big)$ is continuous up to $\mathcal{G}_2^\circ$ and $\mathcal{G}_3$ and, similarly to the proof of Lemma 1.5.3, $W_1\big(G_{ij}(z,z+xR\hat{Z},\eta)\hat{Z}^i\hat{Z}^j\big)^{-n/2}$ is continuous in the support of $\chi\big(P\big)$ and $\chi'\big(P\big)$. This shows continuity of $W_1\, K_\nabla(\cdot,\cdot)$ away from $\mathcal{G}_{10}$, $\mathcal{G}_{11}$ and $\mathcal{G}_{10}$. Now exactly as in the proof of Lemma 1.5.3, the expressions $e^{-\sigma\frac{R\hat{X}}{1+xR\hat{X}}}\chi'(P)$, $e^{-\sigma\frac{R\hat{X}}{1+xR\hat{X}}}\chi(P)$ vanish to infinite order at those three faces, uniformly in $\eta$. All other factors of $W_1K_\nabla$ grow at most polynomially at those faces, thus $W_1K_\nabla \in C^0\big(O^2_{\Delta_s} \times [0,\eta_0)\big)$. This yields the claim for $W_1 \in \mathcal{W}_1$, with $\kappa_{W_1} \equiv 0$ in (1.5.22).

Now suppose $W_2 \in \mathcal{W}_2^J$ and we will again first look away from the faces $\mathcal{G}_{10}$, $\mathcal{G}_{11}$ and $\mathcal{G}_{10}$. If $j \neq J$ we have that

$$R^{-1}\partial_{\theta^j}\Lambda_0 = \sum_{m=0}^{n}\partial_{v^m}\Lambda_0\,\partial_{\theta^j}\hat{Z}^m$$

is continuous up to $x = 0$ on $\mathcal{U}_J^\pm$, using the chain rule. Note that $\Lambda_0(z,0,0,\eta)$ is independent of $\nabla$. Now as already observed in the proof of Lemma 1.5.3, by (1.5.17) one has that $\beta^*_{\Delta_s}(\chi(P)) = \chi\left(\frac{a_1(z,xR\hat{Z},\eta,\hat{Z})}{x(a_2(z,xR\hat{Z},\eta,\hat{Z}))^{1/2}}\right)$, where $a_j(z,u,\eta,v)$ is $C^1$ in $(z,u,\eta)$ and $C^\infty$ in $v$. Moreover, $\partial_{v^j}(a_1/a_2^{1/2})\big|_{u=0} = \dfrac{\delta^0_j(g_{\alpha\beta}v^\alpha v^\beta) - v^0 g_{j\alpha}v^\alpha}{(g_{\alpha\beta}v^\alpha v^b)^{3/2}}$. Thus in $\mathcal{U}_J^\pm$ we have, for $x, R > 0$ and $j \neq J$,

$$\begin{aligned} R^{-1}\partial_{\theta^j}(\beta^*_{\Delta_s}(\chi(P))) = & R^{-1}\chi'(P)\Big(R\partial_u(a_1/a_2^{1/2})(z,xR\hat{Z},\eta,\hat{Z})\cdot\partial_{\theta^j}\hat{Z} \\ & + x^{-1}\partial_v(a_1/a_2^{1/2})(z,xR\hat{Z},\eta,\hat{Z})\cdot\partial_{\theta^j}\hat{Z}\Big). \end{aligned}$$

Now use Taylor's Theorem for the function $R \mapsto \partial_v(a_1/a_2^{1/2})(z,xR\hat{Z},\eta,\hat{Z})\cdot\partial_{\theta^j}\hat{Z}$ (which is

$C^1$ in the support of $\chi'(P)$) for $x>0$ to find

$$
\begin{aligned}
R^{-1}\partial_{\theta^j}(\beta^*_{\Delta_s}(\chi(P))) =& R^{-1}\chi'(P)\Big(R\partial_u(a_1/a_2^{1/2})(z,xR\hat{Z},\eta,\hat{Z})\cdot\partial_{\theta^j}\hat{Z} \\
&+ x^{-1}\partial_{v^m}(a_1/a_2^{1/2})(z,0,\eta,\hat{Z})\partial_{\theta^j}\hat{Z}^m + R\,b_{k\ell}(z,xR\hat{Z},\eta,\hat{Z})\hat{Z}^k\partial_{\theta^j}\hat{Z}^\ell\Big) \\
=& R^{-1}\chi'(P)\Big(R\partial_u(a_1/a_2^{1/2})(z,xR\hat{Z},\eta,\hat{Z})\cdot\partial_{\theta^j}\hat{Z} \\
&+ x^{-1}\frac{\delta^0_m(g_{\alpha\beta}(z-\overline{\eta})v^\alpha v^\beta) - v^0 g_{m\alpha}(z-\overline{\eta})v^\alpha}{(g_{\alpha\beta}(z-\overline{\eta})v^\alpha v^\beta)^{3/2}}\Big|_{v=\hat{Z}}\partial_{\theta^j}\hat{Z}^m \\
&+ R\,b_{k\ell}(z,xR\hat{Z},\eta,\hat{Z})\hat{Z}^k\partial_{\theta^j}\hat{Z}^\ell\Big);
\end{aligned}
\tag{1.5.23}
$$

here $b_{k\ell}(z,u,\eta,v)$ is $C^0$ in $(z,u,\eta)$ and $C^\infty$ in $v$. Note that on $\mathcal{U}_J^\pm$ and for $j\neq J$

$$
\partial_{\theta^j}\hat{Z}^m = \begin{cases} x^{\delta_{0m}}\delta^m_j, & m\neq J \\ -x^{\delta_{0m}}\theta^j/\theta^m, & m=J \end{cases}.
$$

Therefore, evaluating at $v=\hat{Z}$ in (1.5.23)

$$
R^{-1}\partial_{\theta^j}(\beta^*_{\Delta_s}(\chi(P))) = \chi'(P)R^{-1}\frac{(g_{\alpha\beta}(z-\overline{\eta})\hat{Y}^\alpha\hat{Y}^\beta)\partial_{\theta^j}\hat{X} - \hat{X}g_{m\alpha}(z-\overline{\eta})\hat{Z}^\alpha\partial_{\theta^j}\hat{Z}^m}{(g_{\alpha\beta}(z-\overline{\eta})\hat{Y}^\alpha\hat{Y}^\beta)^{3/2}} + \chi'(P)\Lambda_1,
\tag{1.5.24}
$$

where $\Lambda_1\in C^0(\mathcal{U}_J^\pm)$ in the support of $\chi'(P)$ (as in the proof of Lemma 1.5.3) and bounded as $R\to\infty$. Note that upon multiplying (1.5.24) by $R$ and evaluating at $R=0$, the first term on the right hand side is independent of $\nabla$ and the second one vanishes.

We similarly compute that

$$
R^{-1}\partial_{\theta^j}\left(G_{k\ell}(z,z+xR\hat{Z},\eta)\hat{Z}^k\hat{Z}^\ell\right)^{-\frac{n}{2}} = -R^{-1}\frac{n}{2}|\hat{Z}|_g^{-n-2}\big(2g_{k\ell}(z-\overline{\eta})\hat{Z}^k\partial_{\theta^j}\hat{Z}^\ell\big) + \Lambda_2
$$

with $\Lambda_2\in C^0(\mathcal{U}_J^\pm)$ in the support of $\chi(P)$ and bounded as $R\to\infty$.

As before, the expressions $\chi'(P)e^{-\sigma\frac{R\hat{X}}{1+xR\hat{X}}}$ and $\chi(P)e^{-\sigma\frac{R\hat{X}}{1+xR\hat{X}}}$ decay exponentially fast for $R\to\infty$ and in particular vanish identically for $R\hat{X} << -1$, uniformly in $\eta$, thus on $\mathcal{U}_J^\pm$, $x_3W_2(K_\nabla)$ vanishes to infinite order at $\mathcal{G}_{10}$, $\mathcal{G}_{11}$ and $\mathcal{G}_{01}$, uniformly in $\eta$. By a partition of unity subordinate to the cover $\{\mathcal{U}_J^\pm\}_{J,\pm}$ we find that $x_3W_2(K_\nabla)$ satisfies (1.5.21) and (1.5.22). $\square$

We have shown the regularity results for $\widetilde{A}_{\chi,\eta,\sigma}$, under hypotheses that apply for both $\nabla = \widehat{\nabla}, \overline{\nabla}$. We now wish to analyze the kernel of the difference $\widetilde{E}_{\eta,\sigma}$.

**Lemma 1.5.6.** *Let $W$ be the lift to $\overline{X}^2_{\Delta_s}$ of a scattering vector field acting on the left factor of $\overline{X}^2$, and $x_3$ a defining function for $\mathcal{G}_3$, smooth and non-vanishing up to the other boundary hypersurfaces of $\overline{X}^2_{\Delta_s}$. Then for any sufficiently small neighborhood $O$ of $p$ in $\overline{X}$ there exists $\eta_0 > 0$ such that upon writing $\beta^*_{\Delta_s}(\kappa_{\widetilde{E}_{\eta,\sigma}}) = K_E \cdot \nu$ (where as before $\nu$ is a smooth non-vanishing density on $\overline{X}^2_{\Delta_s}$) one has $x_3^{-1}K_E$, $WK_E \in C^0\big(O^2_{\Delta_s} \times [0,\eta_0)\big)$ and vanishing to infinite order at $\mathcal{G}_{10}$, $\mathcal{G}_{11}$ and $\mathcal{G}_{01}$. Moreover, both $WK_E$ and $x_3^{-1}K_E$ vanish identically on $O$ for $\eta = 0$.*

*Proof.* First observe that Lemmas 1.2.1 and 1.3.1 imply that for $\sigma > 0$ and $\chi_0$ fixed in Proposition 1.4.3 Lemmas 1.5.2, 1.5.3 and 1.5.4 apply to both $\overline{\nabla}$ and $\widehat{\nabla}$, provided $\eta_0$ and $O$ are sufficiently small: one needs $O$ to be small enough that if $\operatorname{supp}(\chi_0) \subset [-M, M]$ then $Mx \leq \min\{\widetilde{C}_{\widehat{\nabla}}, \widetilde{C}_{\overline{\nabla}}\}\sqrt{x}$ in $O$, where $\widetilde{C}_{\widehat{\nabla}}$, $\widetilde{C}_{\overline{\nabla}}$ are the constants of Lemma 1.4.2 corresponding to the two connections. Now we observe that $WK_E$ and $x_3^{-1}K_E \in C^0\big(O^2_{\Delta_s} \times [0,\eta_0)\big)$ and both vanishing to infinite order on $\mathcal{G}_{10}$, $\mathcal{G}_{11}$ and $\mathcal{G}_{01}$. To see this we are using the fact that in both (1.5.20) and (1.5.22) the leading order coefficient at the lifted diagonal $\Delta_s$ does not depend on the choice of connection and hence cancels upon taking the difference $K_{\widehat{\nabla}} - K_{\overline{\nabla}}$ (where $K_{\widehat{\nabla}}$, $K_{\overline{\nabla}}$ are computed using the same density $\nu$). Finally if $\eta = 0$ we have $\widetilde{E}_{0,\sigma} = E_{0,\sigma}$, acting on functions supported in a subset of $\overline{X} = \overline{X}_0 \subset M_e^c$. Since $(\overline{\nabla} - \widehat{\nabla})\big|_{T_z\widetilde{M}\times T_z\widetilde{M}} = 0$ provided $z \notin M_e$ and by construction of $\widehat{A}_{\chi_0,0,\sigma}$ (resp. $\overline{A}_{\chi_0,0,\sigma}$), $K_{\widehat{\nabla}}(\cdot,0)$ (resp. $K_{\overline{\nabla}}(\cdot,0)$) only depends on the connection $\widehat{\nabla}$ (resp. $\overline{\nabla}$) on $\overline{X} \subset M_e^c$, we have $WK_E(\cdot,0)$, $x_3^{-1}K_E(\cdot,0) \equiv 0$ and $E_{0,\sigma} = \widetilde{E}_{0,\sigma} = 0$. □

We finally have:

*Proof of Proposition 1.4.6.* As already mentioned, it suffices to show (1.5.11). Let $O' \subset \overline{X}$ be a small open neighborhood of $p$ in $\overline{X}$ where the results of this section hold and $O$ a neighborhood of $p$ in $\overline{X}$ with $O \subset K \subset O'$, where $K$ compact. For sufficiently small $\eta \geq 0$

we have $\widetilde{U}_\eta = \psi_\eta(U_\eta) \subset\subset O$. Fix $\delta > 0$. We will show that there exists an $\eta_0$ such that if $0 \le \eta < \eta_0$ then for $u, v \in L^2(O)$ with $\operatorname{supp} v \subset \widetilde{U}_\eta$, and $\widetilde{W}_j \in \{x^2\partial_x, x\partial_{y^1}, \dots, x\partial_{y^n}\} \subset \mathcal{V}_{sc}(\overline{X})$ one has

$$|(u, \widetilde{W}_j^k\, \widetilde{E}_{\eta,\sigma} v)| \le \delta \|u\|_{L^2(O)} \|v\|_{L^2(\widetilde{U}_\eta)}, \quad j = 0, \dots, n, \quad k = 0, 1. \tag{1.5.25}$$

This will imply the claim since $\widetilde{W}_j$ span $\mathcal{V}_{sc}(\overline{X})$ on $O'$. Let $\pi_{L;\Delta_s} = \pi_L \circ \beta_{\Delta_s}$, $\pi_{R;\Delta_s} = \pi_R \circ \beta_{\Delta_s}$, where $\pi_L$, $\pi_R$ denote projection onto the left and right factor of $\overline{X}^2$ respectively. By Cauchy-Schwartz inequality and using the notations of Lemma 1.5.6

$$\begin{aligned} \Big|\int_{O^2} (u \otimes v)\kappa_{E_{\eta,\sigma}}\Big|^2 &\le \left(\int_{O^2_{\Delta_s}} \big|(\pi^*_{L;\Delta_s} u) K_E(\cdot, \eta)(\pi^*_{R;\Delta_s} v)\big|\nu\right)^2 \\ &\le \int_{O^2_{\Delta_s}} |(\pi^*_{L;\Delta_s} u)|^2 |K_E(\cdot,\eta)|\nu \cdot \int_{O^2_{\Delta_s}} |K_E(\cdot,\eta)|\, |(\pi^*_{R;\Delta_s} v)|^2 \nu. \end{aligned} \tag{1.5.26}$$

Recall that the "coordinates" (1.5.3) and the analogous ones given by

$$\left(\widetilde{x}, \widetilde{y}, \widetilde{R} = \sqrt{\widetilde{X}^2 + |\widetilde{Y}|^2}, \widetilde{\theta} = (\widetilde{X}, \widetilde{Y})/\widetilde{R}\right), \text{ where } \widetilde{X} = \frac{x - \widetilde{x}}{\widetilde{x}^{\,2}}, \widetilde{Y} = \frac{y - \widetilde{y}}{\widetilde{x}} \tag{1.5.27}$$

identify $O^2_{\Delta_s} \setminus (\mathcal{G}_{10} \cup \mathcal{G}_{11} \cup \mathcal{G}_{01})$ with a subset of $\overline{\mathbb{R}^{n+1}_+} \times [0, \infty) \times \mathbb{S}^n$. By interchanging the roles of $(x, y)$ and $(\widetilde{x}, \widetilde{y})$, Lemma 1.5.1 yields the existence of a non-vanishing $\widetilde{\alpha} \in C^\infty(\overline{X}^2_{\Delta_s})$ such that in terms of (1.5.27) $\nu = \widetilde{\alpha}(\widetilde{R} + 1)^{-1}(2 + \widetilde{x}\widetilde{R}\widetilde{\theta}_0)^n |d\widetilde{x}\, d\widetilde{y}\, d\widetilde{R}\, d\widetilde{\omega}|$ ($d\widetilde{\omega}$ is the volume form with respect to the round metric). Thus

$$\begin{aligned} &\int_{O^2_{\Delta_s}} |(\pi^*_{L;\Delta_s} u)|^2 |K_E|\nu \\ &\qquad = \int |u(x,y)|^2 |K_{E;L}(x, y, R, \theta, \eta)| \frac{(2 + xR\theta_0)^n}{1 + R}\, |dx\, dy\, dR\, d\omega| \end{aligned} \tag{1.5.28}$$

and similarly

$$\begin{aligned} &\int_{O^2_{\Delta_s}} |K_E| |(\pi^*_{R;sc} v)|^2 \nu \\ &\qquad = \int |K_{E;R}(\widetilde{x}, \widetilde{y}, \widetilde{R}, \widetilde{\theta}, \eta)|\, |v(\widetilde{x}, \widetilde{y})|^2 \frac{(2 + \widetilde{x}\widetilde{R}\widetilde{\theta}_0)^n}{1 + \widetilde{R}} |d\widetilde{x}\, d\widetilde{y}\, d\widetilde{R}\, d\widetilde{\omega}|, \end{aligned} \tag{1.5.29}$$

where $K_{E;L}$, $K_{E;R}$ express $K_E$ in terms of (1.5.3) and (1.5.27) respectively and the integrations on the right hand sides of (1.5.28) and (1.5.29) are over the appropriate subsets of $\overline{\mathbb{R}^{n+1}_+} \times [0,\infty) \times \mathbb{S}^n$ corresponding to $O^2_{\Delta_s}$ (the function $\widetilde{\alpha}$ has been absorbed into $K_{E;R}$).

By (1.5.4), $(2 + xR\theta_0)^n$ and $(1 + R)$ are of the form $x_{10}^{-n}$ and $x_{11}^{-1}x_{10}^{-2}$ respectively. Letting $M_0$ and $R_0$ be large positive integers we write

$$\begin{aligned}
&\int |u(x,y)|^2 |K_{E;L}(x,y,R,\theta,\eta)| \frac{(2 + xR\hat{X})^n}{1 + R} |dx\, dy\, dR\, d\omega| \\
&\qquad \leq \|u\|^2_{L^2(O)} \sup_{(x,y)\in O} \int_{\mathbb{S}^n} \int_0^\infty |K_{E;L}(x,y,R,\theta,\eta)| \frac{(2 + xR\hat{X})^n}{1 + R} |dR\, d\omega| \\
&\qquad = \|u\|^2_{L^2(O)} \sup_{(x,y)\in O} \Bigg( \int_{\mathbb{S}^n} \int_0^{R_0} |K_{E;L}(x,y,R,\theta,\eta)| \frac{(2 + xR\hat{X})^n}{1 + R} |dR\, d\omega| \\
&\qquad\quad + \int_{\mathbb{S}^n} \int_{R_0}^\infty (1 + R)^{-M_0} \Big\{ (1 + R)^{M_0 - 1} |K_{E;L}(x,y,R,\theta,\eta)| (2 + xR\hat{X})^n \Big\} |dR\, d\omega| \Bigg) \\
&\qquad = \|u\|^2_{L^2(O)} \sup_{(x,y)\in O} \left( \mathrm{I}(x,y,\eta) + \mathrm{II}(x,y,\eta) \right).
\end{aligned}$$

By Lemma 1.5.6, since $K_E$ vanishes to infinite order at $\mathcal{G}_{10}$, $\mathcal{G}_{11}$ and $\mathcal{G}_{01}$, there exists a constant $C$ such that for all $(x,y) \in O$ and all $0 \leq \eta \leq \eta_0$

$$(1 + R)^{M_0 - 1} |K_{E;L}(x,y,R,\theta,\eta)| (2 + xR\hat{X})^n \leq C.$$

Therefore, for given $\delta > 0$, $R_0$ can be chosen sufficiently large that $\mathrm{II}(x,y,\eta) \leq \delta/2$ for $0 \leq \eta \leq \eta_0$. On the other hand, $\mathrm{I}(x,y,\eta)$ is continuous (it is an integral over a compact set of a function continuous jointly in $(x,y,R,\theta,\eta)$) and it vanishes identically for $(x,y,\eta) \in O \times \{0\} \subset K \times \{0\}$ by Lemma 1.5.6. Thus there exists $\eta_0$ such that for $0 \leq \eta \leq \eta_0$ we have $\sup_{(x,y)\in O} \mathrm{I}(x,y,\eta) \leq \delta/2$ and (1.5.28) is bounded above by $\delta\|u\|^2_{L^2(O)}$.

Now (1.5.29) can be analyzed in exactly the same way as (1.5.28); the only difference is that now $(2 + \widetilde{x}\widetilde{R}\widetilde{\theta}_0)^n$ and $(1 + \widetilde{R})$ are of the form $x_{01}^{-n}$ and $x_{11}^{-1}x_{01}^{-2}$. This however does not change the arguments since $K_E$ vanishes to infinite order on $\mathcal{G}_{11}$, $\mathcal{G}_{10}$ and $\mathcal{G}_{01}$ uniformly for small $\eta$. We conclude that (1.5.25) holds for $k = 0$.

To show (1.5.25) for $k = 1$ we observe the following: working similarly to the proof of Lemma 1.5.1 one sees that for $a = x_{11}x_2^{n+2}x_3^n$ it is the case that $a^{-1}\beta^*_{\Delta_s}(|dzd\widetilde{z}|)$ extends from

$(\overline{X}^2_{\Delta_s})^\circ$ to a smooth non-vanishing density $\nu$ on $\overline{X}^2_{\Delta_s}$ (as before, $x_*$ stands for a boundary defining function of $\mathcal{G}_*$ that is smooth and non-vanishing up to the other faces). By the analysis at the end of Section 1.5.1 it follows that for $j = 0, \ldots, n$ the vector field $x_3 W_j$, where $W_j$ is the lift of $\widetilde{W}_j$, is smooth on $\overline{X}^2_{\Delta_s}$ and tangent to all of its boundary hypersurfaces. Thus $(W_j a)/a \in x_3^{-1} C^\infty(\overline{X}^2_{\Delta_s})$. Writing $\kappa_{E_{\eta,\sigma}} = \widetilde{\kappa}_E(z, \widetilde{z}, \eta)|dzd\widetilde{z}|$ so that $\beta^*_{\Delta_s}(\widetilde{\kappa}_E)a = K_E$ we have, for $u$, $v \in L^2(O)$ as before,

$$\begin{aligned}\int_{O^2} u(z)(\widetilde{W}_j\ \widetilde{\kappa}_E(z, \widetilde{z}, \eta))v(\widetilde{z})|dzd\widetilde{z}| &= \int_{O^2_{\Delta_s}} (\pi^*_{L;\Delta_s} u)\, \beta^*_{\Delta_s}(\widetilde{W}_j\, \widetilde{\kappa}_E)(\pi^*_{R;\Delta_s} v)\, a\nu \\ &= \int_{O^2_{\Delta_s}} (\pi^*_{L;\Delta_s} u)\, [W_j \beta^*_{\Delta_s}(\widetilde{\kappa}_E)] a (\pi^*_{R;\Delta_s} v)\, \nu \\ &= \int_{O^2_{\Delta_s}} \big(\pi^*_{L;\Delta_s} u)\, (W_j\, K_E - K_E \frac{W_j\, a}{a}\big)(\pi^*_{R;\Delta_s} v)\, \nu.\end{aligned}$$

Then (1.5.25) for $k = 1$ follows exactly the same steps as for $k = 0$ from (1.5.26) onwards, with $K_E$ replaced by $W_j\, K_E - ((W_j\, a)/a)K_E$: by Lemma 1.5.6 $W_j\, K_E - ((W_j\, a)/a)K_E \in C^0\big(O^2_{\Delta_s} \times [0, \eta_0)\big)$, it vanishes to infinite order on $\mathcal{G}_{10}$, $\mathcal{G}_{11}$, $\mathcal{G}_{01}$ and is identically 0 for $\eta = 0$. This finishes the proof of the proposition. $\square$

# Chapter 2

## ASYMPTOTICALLY HYPERBOLIC MANIFOLDS WITH BOUNDARY CONJUGATE POINTS BUT NO INTERIOR CONJUGATE POINTS[1]

As already discussed in more detail in the Introduction, this chapter is dedicated to the construction of non-trapping asymptotically hyperbolic manifolds with boundary conjugate points but no interior conjugate points. This is done by first constructing a piecewise smooth $C^{1.1}$ metric on $\mathbb{R}^{n+1}$ that compactifies to an AH metric on the ball that satisfies all of the required properties except smoothness. That those properties hold can be shown in this case using explicit formulas for geodesics and Jacobi fields. As a second step, the metric is approximated by smooth metrics while ensuring that none of the properties we need is violated. The chapter is organized as follows. The $C^{1,1}$ metric is constructed in Section 2.1: in 2.1.1 we define it and state some general properties, in 2.1.2 we show explicit formulas for the curvature along geodesics and in 2.1.3 we compute formulas for Jacobi fields and show Theorem 2 in the $C^{1,1}$ case. In Section 2.2 we prove Theorem 2 in the $C^{\infty}$ case. We first reduce Theorem 2 to three propositions (2.2.1, 2.2.2, 2.2.3) concerning stable Jacobi fields and absence of conjugate points for the approximating metrics. Then we carry out the analysis of the derivatives of the stable solutions, prove Proposition 2.2.18 which rules out interior conjugate points, and conclude by proving Propositions 2.2.1, 2.2.2, 2.2.3.

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[1]The material in this chapter has been accepted and is soon to be published: Nikolas Eptaminitakis and C. Robin Graham, *Asymptotically hyperbolic manifolds with boundary conjugate points but no interior conjugate points*, Journal of Geometric Analysis, Springer New York.

## 2.1 The $C^{1,1}$ Metric

### 2.1.1 The Metric

We will construct metrics on $\mathbb{R}^{n+1}\backslash\{0\} \simeq (0,\infty)_\rho \times \mathbb{S}^n$ of the form

$$g = d\rho^2 + \mathcal{A}^2(\rho)\mathring{g} \tag{2.1.1}$$

in polar coordinates that extend smoothly to the origin. Here $\mathring{g}$ denotes the round metric on $\mathbb{S}^n$ and $\mathcal{A}(\rho)$ is a positive function on $(0,\infty)$ to be chosen appropriately, with $\mathcal{A}(\rho) = \sin(\rho)$ for $\rho$ small. Hence in a neighborhood of the origin $g$ is smooth and is isometric to the round metric on $\mathbb{S}^{n+1}$. Relative to the product decomposition $\mathbb{R}^{n+1}\backslash\{0\} \simeq (0,\infty)\times\mathbb{S}^n$, the non-zero Christoffel symbols of $g$ are

$$\Gamma^0_{\alpha\beta} = -\mathcal{A}(\rho)\mathcal{A}'(\rho)\mathring{g}_{\alpha\beta}, \quad \Gamma^\gamma_{\alpha 0} = \mathcal{A}^{-1}(\rho)\mathcal{A}'(\rho)\delta^\gamma_\alpha, \quad \Gamma^\gamma_{\alpha\beta} = \mathring{\Gamma}^\gamma_{\alpha\beta}, \tag{2.1.2}$$

where $\mathring{\Gamma}$ are the Christoffel symbols of the round metric and $\rho$ is the 0-th coordinate.[2] The form of the Christoffel symbols implies that for any $k = 1,\ldots,n+1$, $k$-dimensional Euclidean planes passing through the origin are totally geodesic. To see this, note that the intersection of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with any $k$-dimensional plane through the origin is totally geodesic for the round metric, and that in general an embedded submanifold $M^k \subset \widetilde{M}^d$ is totally geodesic if and only if in any coordinate chart $(U,\varphi)$ for which $\varphi(U\cap M) \subset \{(z,z') \in \mathbb{R}^k\times\mathbb{R}^{d-k} : z' = 0\}$, the Christoffel symbols satisfy $\Gamma^m_{ij} = 0$ on $M\cap U$ for $i,j \leq k$ and all $m \geq k+1$. As a special case, lines of the form $\gamma(t) = tv$ for $v \in \mathbb{R}^{n+1}$ with Euclidean length 1 are totally geodesic, and in fact they are radial unit speed geodesics for $g$.

The curvature tensor of warped product metrics like $g$ can be described as follows. This is a special case of Proposition 42, Chapter 7 in [ONe83].

**Proposition 2.1.1.** *Let $g = d\rho^2 + \mathcal{A}^2(\rho)b$, where $\rho \in \mathbb{R}$, $0 < \mathcal{A} \in C^\infty(\mathbb{R})$, and $b$ is a metric on a manifold $B$. If $R$, $R_b$ denote the Riemannian curvature tensors of $g$, $b$, respectively, and $U$, $V$, $W \in \mathfrak{X}(B)$, then*

[2] Throughout this chapter, Greek indices run from 1 to $n$ and Latin indices run from 0 to $n$.

*(1)* $R(\partial_\rho, V)\partial_\rho = -\mathcal{A}^{-1}(\rho)\mathcal{A}''(\rho)V$

*(2)* $R(V, W)\partial_\rho = 0$

*(3)* $R(\partial_\rho, V)W = \langle V, W\rangle_g \mathcal{A}^{-1}(\rho)\mathcal{A}''(\rho)\partial_\rho$

*(4)* $R(V, W)U = R_b(V, W)U - (\mathcal{A}'(\rho))^2\mathcal{A}^{-2}(\rho)(\langle V, U\rangle_g W - \langle W, U\rangle_g V)$.

For an $O(n+1)$-invariant metric on $\mathbb{R}^{n+1}$, the sectional curvature of a 2-plane $\Pi \subset T_p\mathbb{R}^{n+1}$ at a point $p = \rho\omega$, $\rho > 0$, $\omega \in \mathbb{S}^n$, depends only on $\rho$ and the angle $\alpha$ between $\partial_\rho$ and $\Pi$. We will denote any such plane by $\Pi_{\rho;\cos(\alpha)}$ and the corresponding sectional curvature by $\mathrm{Sec}(\Pi_{\rho;\cos(\alpha)})$.[3] Then by Proposition 2.1.1 we find, for 2-planes parallel to the radial direction,

$$\mathrm{Sec}(\Pi_{\rho;1}) = -\mathcal{A}(\rho)^{-1}\mathcal{A}''(\rho) =: K^{\parallel}(\rho). \tag{2.1.3}$$

Moreover, if $n \geq 2$, for 2-planes normal to the radial direction we have

$$\mathrm{Sec}(\Pi_{\rho;0}) = \mathcal{A}^{-2}(\rho) - \mathcal{A}^{-2}(\rho)(\mathcal{A}'(\rho))^2 =: K^{\perp}(\rho). \tag{2.1.4}$$

More generally, it follows from (2) in Proposition 2.1.1 and the symmetries of the curvature tensor that $R(u, w, u, \partial_\rho) = 0$ for $u$, $w \in \partial_\rho^\perp$, so for $\alpha \in [0, \pi/2]$ we have

$$\mathrm{Sec}(\Pi_{\rho;\cos(\alpha)}) = \cos^2(\alpha)K^{\parallel}(\rho) + \sin^2(\alpha)K^{\perp}(\rho). \tag{2.1.5}$$

It will later be convenient to use (2.1.5) to *define* $\mathrm{Sec}(\Pi_{\rho;\cos(\alpha)})$ for $\cos(\alpha) \in [-1, 0)$ so that (2.1.5) holds for all $\cos(\alpha) \in [-1, 1]$ and $\rho > 0$. From (2.1.3) and the fact that $\mathcal{A}(\rho) = \sin(\rho)$ for small $\rho$ it follows that $\mathcal{A}$ solves the equation $\mathcal{A}''(\rho) + \mathrm{Sec}(\Pi_{\rho;1})\mathcal{A}(\rho) = 0$ with $\mathcal{A}(0) = 0$ and $\mathcal{A}'(0) = 1$.

The previous discussion indicates that the geometry induced by $g$ on $\mathbb{R}^{n+1}$ is entirely determined by the radial curvature function $K^{\parallel}$, thus our goal will be to choose it appropriately.

[3] We use Sec for sectional curvature, as opposed to sec which will be used for the secant of a real number.

We let, for $\rho \geq 0$, $r > 0$ and $\varepsilon \geq 0$,

$$K^{\|}_{r,\varepsilon}(\rho) = \begin{cases} 1 - 2\varphi(\frac{\rho - r}{\varepsilon}) & \varepsilon > 0 \\ 1 - 2H(\rho - r) & \varepsilon = 0, \end{cases}$$

where $\varphi \in C^\infty(\mathbb{R})$ satisfies $0 \leq \varphi \leq 1$, $\varphi(\rho) = 0$ for $\rho \leq 0$ and $\varphi(\rho) = 1$ for $\rho \geq 1$, and $H$ is the Heaviside function: $H(\rho) = 0$ if $\rho \leq 0$, $H(\rho) = 1$ if $\rho > 0$. In particular, $K^{\|}_{r,\varepsilon}(\rho) = 1$ for $\rho \leq r$. Observe that $K^{\|}_{r,\varepsilon}$ is $C^\infty$ if $\varepsilon > 0$ and is piecewise $C^\infty$ if $\varepsilon = 0$. Moreover, for each $r$, $K^{\|}_{r,\varepsilon} - K^{\|}_{r,0} \to 0$ in $L^1([0, \infty))$ as $\varepsilon \to 0$.

Define $\mathcal{A}_{r,\varepsilon}$ to be the solution to

$$\mathcal{A}'' + K^{\|}_{r,\varepsilon}\mathcal{A} = 0, \qquad \mathcal{A}(0) = 0, \quad \mathcal{A}'(0) = 1, \tag{2.1.6}$$

where if $\varepsilon = 0$, $\mathcal{A}_{r,0}$ is interpreted as a weak solution. This means that it is the unique $C^1$ function satisfying the initial conditions in (2.1.6) which in addition solves the differential equation in the open intervals where $K^{\|}_{r,0}$ is smooth. Observe that for all $\varepsilon \geq 0$,

$$\mathcal{A}_{r,\varepsilon}(\rho) = \begin{cases} \sin(\rho) & \rho \leq r \\ a_+ e^{\rho} + a_- e^{-\rho} & \rho \geq r + \varepsilon, \end{cases} \tag{2.1.7}$$

where $a_\pm$ depend on $r$, $\varepsilon$. When $\varepsilon = 0$, the values of $a_\pm$ are determined by matching the value and the derivative at $\rho = r$ with those of $\sin(\rho)$. The case $r = \pi/4$ is special in that $a_- = 0$:

$$\mathcal{A}_{\pi/4,0}(\rho) = \frac{\sqrt{2}}{2} e^{\rho - \pi/4} \qquad \rho \geq \pi/4. \tag{2.1.8}$$

This is fortuitous, because as we will see, $r = \pi/4$ is precisely the value for which the corresponding metric has boundary conjugate points but no interior conjugate points. It is not true that $a_- = 0$ for other choices of $r$, including the degenerate case $r = 0$, $\varepsilon = 0$, which corresponds to hyperbolic space.

The Sturm Comparison Theorem implies that $\mathcal{A}_{r,\varepsilon} > 0$ on $(0, \pi)$ and comparison of the Prüfer angle (Theorem 1.2, p. 210 of [CL55]) shows that $\mathcal{A}'_{r,\varepsilon} > 0$ on $(0, \pi/2)$. Since $\mathcal{A}''_{r,\varepsilon} = \mathcal{A}_{r,\varepsilon}$ on $(r + \varepsilon, \infty)$, it follows that $\mathcal{A}_{r,\varepsilon} > 0$ and $\mathcal{A}'_{r,\varepsilon} > 0$ on $(0, \infty)$ if we require

$r+\varepsilon<\pi/2$, which we do henceforth. In particular, $a_+>0$ in (2.1.7). Ultimately we will only care about $r$ near $\pi/4$ and $\varepsilon$ near 0.

We will denote by $g_{r,\varepsilon}$ the metric given by (2.1.1) with $\mathcal{A}$ replaced by $\mathcal{A}_{r,\varepsilon}$. So $g_{r,\varepsilon}$ restricted to the geodesic ball $B_r(0)$ centered at the origin is isometric to the corresponding geodesic ball in $\mathbb{S}^{n+1}$, and in particular the sectional curvatures of $g_{r,\varepsilon}$ are all equal to 1 for $\rho<r$. The sectional curvature of 2-planes parallel to the radial direction is $-1$ for $\rho>r+\varepsilon$, but not for other 2-planes if $n\geq 2$. The metric $g_{r,\varepsilon}$ is asymptotically hyperbolic (but only $C^{1,1}$ if $\varepsilon=0$) if $\mathbb{R}^{n+1}$ is radially compactified with defining function $e^{-\rho}$ for the boundary at infinity. In particular, $g_{r,\varepsilon}$ is complete.

Our goal in Section 2.1 is to show that $g_{\pi/4,0}$ satisfies all of the properties stated in Theorem 2 except for smoothness.

### *2.1.2 Geodesics and Sectional Curvature*

Since $g=g_{r,\varepsilon}$ is at least $C^{1,1}$, it determines geodesics of class at least $C^{2,1}$. Provided a unit speed geodesic $\gamma(t)$ of $g$ is not radial, $\gamma(0)$ and $\gamma'(0)$ determine a unique 2-plane through the origin denoted by $\Sigma_\gamma$; as mentioned earlier, $\Sigma_\gamma$ is totally geodesic and hence $\gamma$ is entirely contained in it. For radial geodesics $\gamma$, we will write $\Sigma_\gamma$ for any 2-plane containing $\gamma$.

To study any unit speed geodesic $\gamma$ it is sufficient to work in $\Sigma_\gamma$ with induced metric $g\big|_{\Sigma_\gamma}=d\rho^2+\mathcal{A}^2(\rho)d\theta^2$, where $\mathcal{A}=\mathcal{A}_{r,\varepsilon}$. According to (2.1.2), $\rho(t):=\rho(\gamma(t))$ satisfies the equation

$$\rho''=\mathcal{A}^{-1}(\rho)\mathcal{A}'(\rho)\left(1-(\rho')^2\right). \tag{2.1.9}$$

If $\gamma$ is not radial, the initial conditions take the form $\rho(0)=s>0$, $\rho'(0)=v$ with $|v|<1$. It is evident from (2.1.7) that there is $a=a_{r,\varepsilon}>0$ so that $\mathcal{A}^{-1}(\rho)\mathcal{A}'(\rho)\geq a$ for all $\rho>0$. A comparison theorem (e.g. Theorem 11.XVI of [Wal98]) implies that $\rho(t)\geq\overline{\rho}(t)$ for all $t\in\mathbb{R}$, where $\overline{\rho}$ is the solution of

$$\rho''=a\left(1-(\rho')^2\right) \tag{2.1.10}$$

satisfying the same initial conditions. Equation (2.1.10) is separable for $\rho'$; the solution is

$$\overline{\rho}(t) = s + a^{-1}\log\left(\tfrac{1}{2}\big((1+v)e^{at} + (1-v)e^{-at}\big)\right). \tag{2.1.11}$$

It follows in particular that $\rho(t) \to \infty$ as $t \to \pm\infty$ so that $g_{r,\varepsilon}$ is nontrapping. Since $\rho'' > 0$, $\rho$ achieves its minimum at a unique time which we take to be $t = 0$. The corresponding point is the closest point on $\gamma$ to the origin, whose distance to the origin we write $s$. We denote this solution by $\rho_{s,r,\varepsilon}$; it is thus the solution to (2.1.9) with $\mathcal{A} = \mathcal{A}_{r,\varepsilon}$ and with initial conditions $\rho(0) = s > 0$, $\rho'(0) = 0$. For a radial geodesic the distance to the origin is $s = 0$ and the corresponding solution is $\rho_{0,r,\varepsilon}(t) = t$. We denote by $\gamma_{s,r,\varepsilon}$ any unit speed geodesic with radial coordinate function $\rho_{s,r,\varepsilon}$.

If $s < r$, then $\gamma_{s,r,\varepsilon}$ intersects the geodesic ball $B_r(0)$ where the curvature is 1 and $\mathcal{A}(\rho) = \sin(\rho)$. In this case, it is easily checked by directly verifying (2.1.9) and the initial conditions that

$$\rho_{s,r,\varepsilon}(t) = \arccos\big(\cos(s)\cos(t)\big). \tag{2.1.12}$$

This holds up to the time $t$ such that $\rho_{s,r,\varepsilon}(t) = r$. We denote this time by $\ell_r(s)$; geometrically this is the distance between $\gamma_{s,r,\varepsilon}(0)$ and $\partial B_r(0)$ and clearly it is given by

$$\ell_r(s) = \arccos\left(\frac{\cos(r)}{\cos(s)}\right). \tag{2.1.13}$$

For future reference note that

$$\rho'_{s,r,\varepsilon}(\ell_r(s)) = \frac{\sqrt{\cos^2(s) - \cos^2(r)}}{\sin(r)}. \tag{2.1.14}$$

This also has a geometric interpretation: since $\partial_\rho$ and $\gamma'$ are unit vectors, $\rho'_{s,r,\varepsilon}(\ell_r(s)) = \langle\gamma'(\ell_r(s)), \partial_\rho\rangle = \cos(\alpha)$, where $\alpha$ is the the angle between $\gamma'(t)$ and $\partial_\rho$ when $t = \ell_r(s)$, i.e. where $\rho(t) = r$. The above formulas for $\rho_{s,r,\varepsilon}(t)$, $\ell_r(s)$ and $\rho'_{s,r,\varepsilon}(\ell_r(s))$ can also be derived directly via the geometry of $\mathbb{S}^2$.

Our primary focus in Section 2.1 is the case $r = \pi/4$, $\varepsilon = 0$. We suppress these subscripts, so for instance subsequently we write $g = g_{\pi/4,0}$, $\gamma_s = \gamma_{s,\pi/4,0}$, $\rho_s(t) = \rho_{s,\pi/4,0}(t) = \rho(\gamma_s(t))$, $\ell(s) = \ell_{\pi/4}(s)$. Note that for $r = \pi/4$, (2.1.14) reduces to $\rho'_s(\ell(s)) = \sqrt{\cos(2s)}$.

When $r = \pi/4$ and $\varepsilon = 0$, (2.1.8) shows that (2.1.9) for $\rho > \pi/4$ reduces to (2.1.10) with $a = 1$. The initial conditions for $\rho_s$ are $\rho_s(\ell(s)) = \pi/4$, $\rho_s'(\ell(s)) = \sqrt{\cos(2s)}$. The solution is given by (2.1.11) with $t$ replaced by $t - \ell(s)$. It can be written in the form

$$\rho_s(t) = \pi/4 + \log F(t,s) \qquad t \geq \ell(s),$$

where

$$F(t,s) = \cosh(t - \ell(s)) + \sqrt{\cos(2s)}\sinh(t - \ell(s)). \tag{2.1.15}$$

For $t \leq -\ell(s)$ one has $\rho_s(t) = \rho_s(-t)$.

Equation (2.1.5) expresses the sectional curvatures of $g$ in terms of the distance $\rho$ to the origin and the angle $\alpha$ between $\partial_\rho$ and the plane $\Pi$. In our subsequent analysis of Jacobi fields, the sectional curvature for $g_{r,\varepsilon}$ along a geodesic $\gamma_{s,r,\varepsilon}(t)$ of the plane spanned by $\gamma_{s,r,\varepsilon}'(t)$ and a vector normal to $\Sigma_{\gamma_{s,r,\varepsilon}}$ will play a fundamental role. Since $\rho_{s,r,\varepsilon}'(t) = \cos(\angle(\gamma_{s,r,\varepsilon}'(t), \partial_\rho))$, the sectional curvature of interest is

$$K_{s,r,\varepsilon}(t) := \mathrm{Sec}(\Pi_{\rho_{s,r,\varepsilon}(t);\rho_{s,r,\varepsilon}'(t)}).$$

As usual we write $K_s = K_{s,\pi/4,0}$.

**Lemma 2.1.2.** *Suppose $n \geq 2$ and $0 \leq s < \pi/4$. Then*

$$K_s(t) = \begin{cases} 1, & 0 \leq |t| < \ell(s) \\ -1 + 4\sin^2(s)F^{-4}(|t|,s), & |t| > \ell(s) \end{cases}$$

*Proof.* For $\rho > \pi/4$ we have $K^{\|}(\rho) = -1$ and $K^{\perp}(\rho) = -1 + 2e^{-2(\rho - \pi/4)}$. Hence, for $|t| > \ell(s)$ (2.1.5) yields

$$\begin{aligned} \mathrm{Sec}(\Pi_{\rho_s(t);\rho_s'(t)}) &= (\rho_s'(t))^2(-1) + (1 - (\rho_s'(t))^2)\left(-1 + 2e^{-2(\rho_s(t) - \pi/4)}\right) \\ &= -1 + 4\sin^2(s)F^{-4}(|t|,s). \end{aligned} \tag{2.1.16}$$

$\square$

There is a similar analysis for geodesics that do not intersect the geodesic ball $B_{\pi/4}(0)$. This time (2.1.8) holds along the whole geodesic, and the solution (2.1.11) of the geodesic equation satisfying the initial conditions $\rho(0) = s$, $\rho'(0) = 0$ is $\rho_s(t) = s + \log(\cosh(t))$. Repeating the computation (2.1.16) yields the following.

**Lemma 2.1.3.** *Suppose $n \geq 2$ and $s \geq \pi/4$. Then*

$$K_s(t) = -1 + 2e^{-2s+\pi/2}\operatorname{sech}^4(t), \quad t \in \mathbb{R}.$$

In the analysis above we have only used formulas for the radial coordinate of geodesics. We summarize them here and for completeness also provide the angular coordinate $\theta$ for a geodesic, even though it will not play a role in the rest of this chapter. With $F(t,s)$ as in (2.1.15),

$$\rho_s(t) = \begin{cases} \arccos(\cos(t)\cos(s)), & 0 \leq s < \pi/4,\ |t| \leq \ell(s) \\ \log\left(F(|t|,s)\right) + \pi/4, & 0 \leq s < \pi/4,\ |t| > \ell(s) \\ \log(\cosh(t)) + s, & s \geq \pi/4,\ t \in \mathbb{R} \end{cases}.$$

Setting

$$\theta_s(t) := \begin{cases} \arcsin\left(\dfrac{\sin(t)}{\sqrt{1-\cos^2(s)\cos^2(t)}}\right), & 0 \leq s < \pi/4, \\ & |t| \leq \ell(s) \\ \operatorname{sgn}(t)\left(\dfrac{2\sin(s)\sinh\left(|t| - \ell(s)\right)}{F(|t|,s)} + \arcsin\left(\sqrt{1-\tan^2(s)}\right)\right), & 0 \leq s < \pi/4, \\ & |t| > \ell(s) \\ \sqrt{2}\tanh(t)e^{-s+\pi/4}, & s \geq \pi/4, \\ & t \in \mathbb{R} \end{cases},$$

the curve $(\rho_s(t), \theta_s(t))$ on $\Sigma_{\gamma_s}$ satisfies the geodesic equation for each $s \geq 0$. Any other geodesic on $\Sigma_{\gamma_s}$ can be obtained by translation in $\theta$.

### *2.1.3 Analysis of Jacobi Fields*

In this subsection we first identify the scalar equations solved by normal Jacobi fields for $g_{r,\varepsilon}$. Then we compute explicitly the normal Jacobi fields of the $C^{1,1}$ metric $g = g_{\pi/4,0}$ and show that $(\mathbb{R}^{n+1}, g)$ has no interior conjugate points but has boundary conjugate points.

The following general fact can be proved using the Gauss and Codazzi equations:

**Proposition 2.1.4.** *Let $(M, g_M)$ be a totally geodesic submanifold of a Riemannian manifold $(\widetilde{M}, g_{\widetilde{M}})$. Let $\gamma$ be a geodesic contained in $M$ and $Y$ be a normal $g_{\widetilde{M}}$-Jacobi field along $\gamma$. When $Y$ is decomposed as $Y = Y_1 + Y_2$, where $Y_1$ is everywhere tangent and $Y_2$ is everywhere normal to $M$, then $Y_2$ is a Jacobi field in $\widetilde{M}$ and $Y_1$ is a Jacobi field in both $M$ and $\widetilde{M}$.*

Proposition 2.1.4 implies that to analyze normal Jacobi fields along a geodesic $\gamma$ of $g_{r,\varepsilon}$, it is enough to analyze separately Jacobi fields tangent and normal to $\Sigma_\gamma$.

Consider first a geodesic $\gamma \subset \Sigma_\gamma$ and a Jacobi field $Y(t)$ normal to $\gamma$ but tangent to $\Sigma_\gamma$ of the form $Y(t) = \mathcal{Y}(t)E(t)$, where $E(t)$ is a parallel vector field along $\gamma$ and $\mathcal{Y}$ is real valued. Since $Y(t)$ is a Jacobi field in the 2-dimensional manifold $\Sigma_\gamma$ and the radial vector field is parallel to $\Sigma_\gamma$, $\mathcal{Y}(t)$ solves the scalar Jacobi equation

$$\mathcal{Y}''(t) + K^{\|}_{r,\varepsilon}(\rho_{s,r,\varepsilon}(t))\mathcal{Y}(t) = 0. \tag{2.1.17}$$

Next consider Jacobi fields along a geodesic $\gamma$ that are orthogonal to the plane $\Sigma_\gamma$. The following lemma reduces the problem to the study of scalar equations.

**Lemma 2.1.5.** *Let $n \geq 2$, $\gamma \subset \Sigma_\gamma$ be a unit speed geodesic for $g_{r,\varepsilon}$ and $Y \perp \Sigma_\gamma$ a Jacobi field along it. Then $Y$ satisfies the scalar Jacobi equation*

$$D_t^2 Y(t) + K_{s,r,\varepsilon}(t)Y(t) = 0.$$

*Proof.* It is sufficient to show that $R(\gamma'(t), Y(t))\gamma'(t) = a(t)Y(t)$, $t \in \mathbb{R}$, for some scalar function $a(t)$; then necessarily $a(t) = K_{s,r,\varepsilon}(t)$, since the plane determined by $\gamma'(t)$ and $Y(t)$ is is of the form $\Pi_{\rho_{s,r,\varepsilon}(t);\rho'_{s,r,\varepsilon}(t)}$. The statement is local, so we can use polar coordinates $(\rho, \theta)$

on $\Sigma_\gamma$ to write $\gamma'(t) = \lambda(t)\partial_\rho + \mu(t)\partial_\theta$. This implies

$$R(\gamma', Y)\gamma' = \lambda^2 R(\partial_\rho, Y)\partial_\rho + \lambda\mu(R(\partial_\rho, Y)\partial_\theta + R(\partial_\theta, Y)\partial_\rho) + \mu^2 R(\partial_\theta, Y)\partial_\theta.$$

By Proposition 2.1.1, for the first term we have $R(\partial_\rho, Y(t))\partial_\rho = -\mathcal{A}''(t)/\mathcal{A}(t)Y(t)$, the second term vanishes and for the third we have

$$R(\partial_\theta, Y(t))\partial_\theta = R_{\mathbb{S}^n}(\partial_\theta, Y(t))\partial_\theta - (\mathcal{A}'(t))^2/\mathcal{A}^2(t)|\partial_\theta|_g^2 Y(t).$$

Now $R_{\mathbb{S}^n}(\partial_\theta, Y(t))\partial_\theta = Y(t)$ since $\mathbb{S}^n$ has constant sectional curvature 1, so the lemma is proved. □

So if we take $Y(t)$ as in Lemma 2.1.5 of the form $Y(t) = \mathcal{Y}(t)E(t)$, where $E(t)$ is a parallel vector field along $\gamma$, then $\mathcal{Y}$ solves the equation

$$\mathcal{Y}''(t) + K_{s,r,\varepsilon}(t)\mathcal{Y}(t) = 0 \tag{2.1.18}$$

where

$$K_{s,r,\varepsilon}(t) = (\rho_\mu'(t))^2 K^{\|}_{r,\varepsilon}(\rho_\mu(t)) + \big(1 - (\rho_\mu'(t))^2\big) K^{\perp}_{r,\varepsilon}(\rho_\mu(t)), \quad \mu = (s, r, \varepsilon) \tag{2.1.19}$$

and $K^{\perp}_{r,\varepsilon}(\rho)$ is given by (2.1.4) with $\mathcal{A}$ replaced by $\mathcal{A}_{r,\varepsilon}$. Note that $K_{s,r,\varepsilon} = K^{\|}_{r,\varepsilon} \circ \rho_{s,r,\varepsilon}$ for $s = 0$. So for radial geodesics the equations (2.1.17) and (2.1.18) for Jacobi fields tangent and normal to $\Sigma_\gamma$ coincide.

We write $\mathcal{U}^{\|}_{s,r,\varepsilon}(t)$, $\mathcal{V}^{\|}_{s,r,\varepsilon}(t)$ for the solutions (weak solutions if $\varepsilon = 0$) of (2.1.17) with the initial conditions $\mathcal{U}^{\|}_{s,r,\varepsilon}(0) = 1$, $\mathcal{U}^{\|}_{s,r,\varepsilon}{}'(0) = 0$ and $\mathcal{V}^{\|}_{s,r,\varepsilon}(0) = 0$, $\mathcal{V}^{\|}_{s,r,\varepsilon}{}'(0) = 1$. Likewise we write $\mathcal{U}^{\perp}_{s,r,\varepsilon}(t)$, $\mathcal{V}^{\perp}_{s,r,\varepsilon}(t)$ for the solutions of (2.1.18) satisfying $\mathcal{U}^{\perp}_{s,r,\varepsilon}(0) = 1$, $\mathcal{U}^{\perp}_{s,r,\varepsilon}{}'(0) = 0$ and $\mathcal{V}^{\perp}_{s,r,\varepsilon}(0) = 0$, $\mathcal{V}^{\perp}_{s,r,\varepsilon}{}'(0) = 1$. And once again we suppress $(r, \varepsilon)$ when $(r, \varepsilon) = (\pi/4, 0)$.

Now we solve (2.1.17), (2.1.18) for $(r, \varepsilon) = (\pi/4, 0)$, beginning with (2.1.17). If $s \geq \pi/4$ then $K^{\|} = -1$, so

$$\mathcal{U}^{\|}_s(t) = \cosh(t), \qquad \mathcal{V}^{\|}_s(t) = \sinh(t) \qquad\qquad s \geq \pi/4.$$

If $s < \pi/4$ then $K^{\|}(\rho_s(t))$ has a jump discontinuity at $|t| = \ell(s)$ and the solutions must be $C^1$ across the jump. It is easily verified that

$$\mathcal{U}_s^{\|}(t) = \begin{cases} \cos(t), & |t| \leq \ell(s) \\ \cos(\ell(s))\cosh(|t| - \ell(s)) - \sin(\ell(s))\sinh(|t| - \ell(s)), & |t| > \ell(s) \end{cases}, \tag{2.1.20}$$

$$\mathcal{V}_s^{\|}(t) = \begin{cases} \sin(t), & |t| \leq \ell(s) \\ \operatorname{sign}(t)\big(\sin(\ell(s))\cosh(|t| - \ell(s)) + \cos(\ell(s))\sinh(|t| - \ell(s))\big), & |t| > \ell(s) \end{cases}. \tag{2.1.21}$$

Recall that $\ell(0) = \pi/4$. So $\mathcal{U}_0^{\|}(t) = \frac{\sqrt{2}}{2}e^{-(|t|-\pi/4)}$ for $|t| > \pi/4$. The corresponding Jacobi field vanishes as $|t| \to \infty$. Hence $g$ has boundary conjugate points along radial geodesics.

**Lemma 2.1.6.** *Let $\gamma$ be a unit speed geodesic for $g_{\pi/4,0}$ contained in a 2-dimensional plane $\Sigma_\gamma$ through the origin. Any non-trivial Jacobi field $Y(t)$ normal to $\gamma$ and tangent to $\Sigma_\gamma$ vanishes at most once.*

*Proof.* We claim that for any $s \geq 0$, $\mathcal{U}_s^{\|}$ is a positive solution of (2.1.17). This is clear when $s \geq \pi/4$ where $\mathcal{U}_s^{\|}(t) = \cosh(t)$. For $s < \pi/4$ it follows from (2.1.20) and the fact that $\sin(\ell(s)) \leq \sqrt{2}/2 \leq \cos(\ell(s))$ (recall (2.1.13)). Now the usual Sturm Separation Theorem is valid for an ODE of the form $\mathcal{Y}''(t) + k(t)\mathcal{Y}(t) = 0$, where $k$ is integrable and real valued and the derivatives are interpreted in a weak sense (see, e.g. comment in [CL55], p. 208). Thus no non-trivial solution of (2.1.17) can vanish twice. $\square$

Recall that $K_s(t)$ is identified in Lemmas 2.1.2 and 2.1.3. We were astonished to find that the scalar Jacobi equations

$$\mathcal{Y}''(t) + K_s(t)\mathcal{Y}(t) = 0 \tag{2.1.22}$$

can be solved explicitly. To do so, note first that for all $t$ if $s \geq \pi/4$ and for $|t| > \ell(s)$ if $s < \pi/4$, $K_s(t)$ has the form $K_s(t) = -1 + f^{-4}(t)$, where $f''(t) - f(t) = 0$. Observing that for any $t_0$ such that $f(t_0) \neq 0$ one has $\left(\frac{\sinh(t-t_0)}{f(t_0)f(t)}\right)' = f^{-2}(t)$, it is easy to check that

$\mathcal{Y}(t) = f(t)b\left(\frac{\sinh(t-t_0)}{f(t_0)f(t)}\right)$ with $b''(x) + b(x) = 0$ is the general solution of the equation $\mathcal{Y}''(t) + (-1 + f^{-4}(t))\mathcal{Y}(t) = 0$.

For each $s > 0$ we identify the solutions $\mathcal{U}_s^\perp$ and $\mathcal{V}_s^\perp$ of (2.1.22). For $s \geq \pi/4$ we take $t_0 = 0$ and obtain

$$\begin{aligned} \mathcal{U}_s^\perp(t) &= \cosh(t)\cos\left(\sqrt{2}e^{-s+\pi/4}\tanh(t)\right) \\ \mathcal{V}_s^\perp(t) &= \frac{\sqrt{2}}{2}e^{s-\pi/4}\cosh(t)\sin\left(\sqrt{2}e^{-s+\pi/4}\tanh(t)\right). \end{aligned} \tag{2.1.23}$$

For $0 < s < \pi/4$ we take $t_0 = \pm\ell(s)$ and obtain

$$\mathcal{U}_s^\perp(t) = \begin{cases} \cos(t), & |t| \leq \ell(s) \\ \frac{\sqrt{2}}{2}\csc(s)F(|t|, s)\cos(\Theta(|t|, s)) & |t| > \ell(s) \end{cases}, \qquad 0 < s < \pi/4, \tag{2.1.24}$$

where $\Theta(t, s) := 2\sin(s)\sinh(t - \ell(s))F^{-1}(t, s) + \arccos(\tan(s))$ and $F(t, s)$ is as in (2.1.15). Note here that $\cos(x + \arccos(\tan(s)))$ is a solution of $b''(x) + b(x) = 0$. Also

$$\mathcal{V}_s^\perp(t) = \begin{cases} \sin(t), & |t| \leq \ell(s) \\ \operatorname{sign}(t)\frac{\sqrt{2}}{2}F(|t|, s)\sin(\Theta(|t|, s)) & |t| > \ell(s) \end{cases}, \quad 0 < s < \pi/4. \tag{2.1.25}$$

We remark that these solutions extend smoothly to $s = 0$ and $\mathcal{U}_0^\perp = \mathcal{U}_0^\parallel$, $\mathcal{V}_0^\perp = \mathcal{V}_0^\parallel$. This is clear for $\mathcal{V}_s$, but for $\mathcal{U}_s$ requires evaluating the indeterminant expression appearing in (2.1.24).

**Lemma 2.1.7.** *Let $n \geq 2$ and $\gamma$ be a unit speed geodesic for $g_{\pi/4,0}$ contained in a 2-dimensional plane $\Sigma_\gamma$ through the origin. Any nontrivial Jacobi field $Y(t)$ along $\gamma$ normal to $\Sigma_\gamma$ vanishes at most once.*

*Proof.* For radial geodesics the proof of Lemma 2.1.6 applies since the equations for Jacobi fields tangent and normal to $\Sigma_\gamma$ coincide.

We claim that $\mathcal{U}_s^\perp$ is everywhere positive for any $s > 0$. For $s \geq \pi/4$ this is clear from (2.1.23) since $|\sqrt{2}e^{-s+\pi/4}\tanh(t)| \leq \sqrt{2} < \pi/2$. For $0 < s < \pi/4$, according to (2.1.24) it suffices to show that $0 < \Theta(t, s) < \pi/2$ for $t \geq \ell(s)$. It is easily verified that for $0 < s < \pi/4$

one has $\partial_t \Theta(t,s) > 0$ for $t \geq \ell(s)$. So for each $s$, $\Theta(t,s)$ strictly increases from a minimum of $\arccos(\tan(s))$ at $t = \ell(s)$ to a limit of

$$\Theta_\infty(s) := \lim_{t\to\infty} \Theta(t,s) = \arccos(\tan(s)) + \frac{2\sin(s)}{1+\sqrt{\cos(2s)}}. \tag{2.1.26}$$

A straightforward calculation shows that

$$\partial_s \Theta_\infty(s) = \frac{-2\sin^2 s}{\cos(s)\left(1+\sqrt{\cos(2s)}\right)^2} \qquad 0 \leq s < \pi/4. \tag{2.1.27}$$

So $\Theta_\infty(s)$ strictly decreases from a maximum of $\pi/2$ at $s = 0$ to a minimum of $\sqrt{2}$ at $s = \pi/4$. Thus $0 < \Theta(t,s) < \pi/2$ for $0 < s < \pi/4$ and $t \geq \ell(s)$.

Once again the result now follows from the Sturm Separation Theorem. □

*Proof of Theorem 2, $C^{1,1}$ metric.* We have already noted that $g = g_{\pi/4,0}$ is non-trapping and has boundary conjugate points along radial geodesics. If $Y$ is a normal Jacobi field along a unit speed geodesic $\gamma \subset \Sigma_\gamma$, write $Y = Y_1 + Y_2$ with $Y_1$ tangent to $\Sigma_\gamma$ and $Y_2$ normal to it, as in Proposition 2.1.4. If $Y$ vanishes twice, so do $Y_1$ and $Y_2$. Lemmas 2.1.6 and 2.1.7 imply that both $Y_1$ and $Y_2$ vanish identically, so $g$ has no interior conjugate points. □

## 2.2 *Smooth Perturbation*

In this section we show that we can find $(r,\varepsilon)$ near $(\pi/4, 0)$ with $\varepsilon > 0$ so that $g_{r,\varepsilon}$ has no interior conjugate points but has boundary conjugate points along radial geodesics, thus proving Theorem 2. First we outline the argument. Our analysis will focus on the decaying (also called stable) solutions of the Jacobi equations (2.1.17), (2.1.18). As we argue below, since $K^{\|}_{r,\varepsilon}(\rho_{s,r,\varepsilon}(t))$ and $K_{s,r,\varepsilon}(t)$ are asymptotic to $-1$ as $t \to \infty$, there are unique solutions $\mathcal{Y}^{\|}_{s,r,\varepsilon}(t)$, $\mathcal{Y}^{\perp}_{s,r,\varepsilon}(t)$ to (2.1.17), (2.1.18), resp., such that $\lim_{t\to\infty} e^t \mathcal{Y}^{\|}_{s,r,\varepsilon}(t) = 1$, $\lim_{t\to\infty} e^t \mathcal{Y}^{\perp}_{s,r,\varepsilon}(t) = 1$. For $K^{\|}_{r,\varepsilon}(\rho_{s,r,\varepsilon}(t))$ this is clear since $K^{\|}_{r,\varepsilon}(\rho) = -1$ for $\rho$ large. Of course for $s = 0$ we have $\mathcal{Y}^{\|}_{0,r,\varepsilon} = \mathcal{Y}^{\perp}_{0,r,\varepsilon}$ since $K^{\|}_{r,\varepsilon} \circ \rho_{0,r,\varepsilon} = K_{0,r,\varepsilon}$. We will show that $\mathcal{Y}^{\|}_{0,r,\varepsilon}(0) \neq 0$ for $(r,\varepsilon)$ sufficiently near $(\pi/4, 0)$. If ${\mathcal{Y}^{\|}_{0,r,\varepsilon}}'(0) = 0$ and $E(t)$ is a non-zero parallel vector field along $\gamma_{0,r,\varepsilon}$, then $\mathcal{Y}^{\|}_{0,r,\varepsilon}(|t|)E(t)$ is a nontrivial Jacobi field which decays as $t \to \pm\infty$. The corresponding metric $g_{r,\varepsilon}$ therefore has boundary conjugate points along radial geodesics. We will prove

**Proposition 2.2.1.** *Any neighborhood of* $(\pi/4, 0)$ *contains a point* $(r, \varepsilon)$ *with* $\varepsilon > 0$ *so that* $\mathcal{Y}^{\|}_{0,r,\varepsilon}{}'(0) = 0$.

It then remains to show that the corresponding metric $g_{r,\varepsilon}$ has no interior conjugate points. We will do this via the following two propositions.

**Proposition 2.2.2.** *There exist a neighborhood* $U$ *of* $(\pi/4, 0)$ *and* $\sigma > 0$ *such that if* $(r, \varepsilon) \in U$ *and* $\mathcal{Y}^{\|}_{0,r,\varepsilon}{}'(0) = 0$*, then* $g_{r,\varepsilon}$ *has no interior conjugate points along any geodesic* $\gamma_{s,r,\varepsilon}$ *with* $0 \leq s \leq \sigma$.

**Proposition 2.2.3.** *For every* $\sigma > 0$*, there exists a neighborhood* $V$ *of* $(\pi/4, 0)$ *so that if* $(r, \varepsilon) \in V$*, then* $g_{r,\varepsilon}$ *has no interior conjugate points along any geodesic* $\gamma_{s,r,\varepsilon}$ *with* $s \geq \sigma$.

Theorem 2 reduces to these three propositions:

*Proof of Theorem 2.* Choose $U$ and $\sigma$ as in Proposition 2.2.2. Then choose $V$ as in Proposition 2.2.3 corresponding to this $\sigma$. Proposition 2.2.1 asserts that there is $(r, \varepsilon) \in U \cap V$ with $\varepsilon > 0$ so that $\mathcal{Y}^{\|}_{0,r,\varepsilon}{}'(0) = 0$. The metric $g_{r,\varepsilon}$ then has boundary conjugate points but no interior conjugate points, and, as before, it is non-trapping. □

Note that by successively shrinking the neighborhoods, one obtains a sequence of metrics $g_{r_j,\varepsilon_j}$ with $\varepsilon_j > 0$ and $(r_j, \varepsilon_j) \to (\pi/4, 0)$ such that each $g_{r_j,\varepsilon_j}$ has boundary conjugate points but no interior conjugate points. The proof actually shows that for each $\varepsilon$ sufficiently small, there is $r_\varepsilon$ so that $g_{r_\varepsilon,\varepsilon}$ has boundary conjugate points but no interior conjugate points.

Continuity as $\varepsilon \to 0$ of solutions of (2.1.17), (2.1.18) and of various of their derivatives in $s$ and $t$ are essential to the proofs of Propositions 2.2.1, 2.2.2, 2.2.3. This is a singular limit, as the functions $K^{\|}_{r,\varepsilon}(\rho_{s,r,\varepsilon}(t))$ and $K_{s,r,\varepsilon}(t)$ develop jump singularities as $\varepsilon \to 0$. We have had to do quite a bit of work to prove the necessary continuity properties. We present this continuity analysis next and afterwards return to the proofs of Propositions 2.2.1, 2.2.2, 2.2.3. We begin the analysis by formulating some general results on ODE: Propositions 2.2.4-2.2.8, that we will apply in our setting.

Let $\mathcal{F} : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field. Suppose that $\mathcal{F}$ is continuous and piecewise $C^\infty$: there is a smooth hypersurface $S \subset \mathbb{R}^d$ locally dividing $\mathbb{R}^d$ into two open subsets $\mathcal{U}_+, \mathcal{U}_-$ and two smooth vector fields $\mathcal{F}_+, \mathcal{F}_-$ on $\mathbb{R}^d$ so that $\mathcal{F} = \mathcal{F}_\pm$ on $\mathcal{U}_\pm$ and $\mathcal{F}_+ = \mathcal{F}_-$ on $S$. Consider the integral curves of $\mathcal{F}$, which solve the ODE

$$x'(t) = \mathcal{F}(x(t)), \qquad x(0) = s.$$

Since $\mathcal{F}$ is Lipschitz, for each $s$ there exists a unique solution $x(s,t)$, and $x(s,t)$ and $x'(s,t)$ are jointly continuous. We assume below that the solutions exist on all the time intervals considered.

**Proposition 2.2.4.** *Suppose that $(s_0, t_0)$ has the property that $x(s_0, 0)$ and $x(s_0, t_0)$ are on opposite sides of $S$, and the curve $t \mapsto x(s_0, t)$, $0 < t < t_0$, crosses $S$ exactly once and does so transversely. There is a smooth function $T(s)$ defined for $s$ in a neighborhood $\mathcal{V}$ of $s_0$ such that $0 < T(s) < t_0$ and $x(s, T(s)) \in S$. The restrictions of $x(s,t)$ to the two sets $\{(s,t) : s \in \mathcal{V},\ 0 \le t \le T(s)\}$ and $\{(s,t) : s \in \mathcal{V},\ T(s) \le t \le t_0\}$ are $C^\infty$.*

*Proof.* Suppose $x(s_0, 0) \in \mathcal{U}_-$ and $x(s_0, t_0) \in \mathcal{U}_+$. The curve $t \mapsto x(s_0, t)$ is an integral curve of $\mathcal{F}_-$ up until the time that $x(s_0, t) \in S$. Since $\mathcal{F}_-$ is smooth, its integral curves are smooth functions of $(s,t)$. Since the crossing is transverse, there is a unique smooth function $T(s)$ defined for $s$ near $s_0$ by the condition that $x(s, T(s)) \in S$. The map $s \mapsto x(s, T(s))$ is smooth from a neighborhood of $s_0$ to $S$, and $x(s,t)$ is smooth for $t \le T(s)$. For $s$ near $s_0$, the curve $t \mapsto x(s,t)$, $t \ge T(s)$ is an integral curve of $\mathcal{F}_+$ whose initial point $x(s, T(s))$ depends smoothly on $s$. By smoothness of the integral curves of $\mathcal{F}_+$, it follows that $x(s,t)$ is smooth for $t \ge T(s)$. □

**Proposition 2.2.5.** *In the setting of Proposition 2.2.4, $x(s,t)$ is jointly $C^1$ in a neighborhood of $(s_0, T(s_0))$.*

*Proof.* We know that $x'(s,t)$ is continuous, and that $\partial_s x(s,t)$ exists on $\{t \ne T(s)\}$ and extends smoothly up to $\{t = T(s)\}$ separately from each side. It suffices to show that the values from the two sides agree on $\{t = T(s)\}$. Let $x_-$ denote the restriction of $x$ to $\{(s,t) : s \in \mathcal{V},\, 0 \le t \le T(s)\}$ and $x_+$ the restriction of $x$ to $\{(s,t) : s \in \mathcal{V},\, T(s) \le t \le t_0\}$. Then $x_+(s, T(s)) = x_-(s, T(s))$ for all $s$. Differentiation in $s$ shows that $\partial_s x_+ + \partial_s T \cdot x'_+ = \partial_s x_- + \partial_s T \cdot x'_-$ when $t = T(s)$. Since $x'_+ = x'_-$ when $t = T(s)$, it follows that also $\partial_s x_+ = \partial_s x_-$ when $t = T(s)$. $\square$

Similar arguments apply for linear equations with piecewise smooth coefficients. In this case we do not assume continuity across the singularity. Consider an intial value problem

$$x'(s,t) = M(s,t)x(s,t), \qquad x(s,0) = x_0(s), \tag{2.2.1}$$

where $x(s,t) \in \mathbb{R}^d$, $M(s,t) \in \mathbb{R}^{d\times d}$, and the parameter $s \in \mathbb{R}^k$. We assume that

$$M(s,t) = \begin{cases} M_-(s,t) & t < T(s) \\ M_+(s,t) & t > T(s), \end{cases}$$

where $T(s) > 0$, $s \mapsto T(s)$ is $C^\infty$, each of $M_\pm$ is $C^\infty$ on $\mathbb{R}^k \times \mathbb{R}$, and $x_0$ is $C^\infty$. It is not assumed that $M_-(s, T(s)) = M_+(s, T(s))$. We require that $x(s,t)$ is a weak solution in the sense that that for each $s$, $x(s,t)$ is a solution for $t \ne T(s)$, and $x$ is continuous across $t = T(s)$. The proof of the following proposition is similar to that of Proposition 2.2.4:

**Proposition 2.2.6.** *The problem* (2.2.1) *has a unique weak solution for each $s$, and the restrictions of $x(s,t)$ to $\{(s,t) : t \le T(s)\}$ and $\{(s,t) : t \ge T(s)\}$ are $C^\infty$.*

*Proof.* There is a unique solution $x_-(s,t)$ to

$$x'_-(s,t) = M_-(s,t)x_-(s,t), \qquad x_-(s,0) = x_0(s),$$

and $x_-$ is $C^\infty$. Likewise, there is a unique solution $x_+(s,t)$ to

$$x'_+(s,t) = M_+(s,t)x_+(s,t), \qquad x_+(s,T(s)) = x_-(s,T(s)),$$

and $x_+$ is $C^\infty$. The function defined by

$$x(s,t) = \begin{cases} x_-(s,t) & t \le T(s) \\ x_+(s,t) & t \ge T(s) \end{cases}$$

is a weak solution of (2.2.1), and is clearly the only weak solution. $\square$

To analyze continuity at $\varepsilon = 0$ we will use the following two results, which are standard applications of Gronwall's inequality. Let $|\cdot|$ denote the Euclidean norm on vectors, or the Euclidean operator norm on matrices.

**Proposition 2.2.7.** *Let $\mathcal{K}_i : [t_0, t_1] \to \mathbb{R}^{d\times d}$, $i = 1, 2$, be bounded and measurable with $|\mathcal{K}_1(t)| \leq L$, and let $f_i : [t_0, t_1] \to \mathbb{R}^d$, $i = 1, 2$, be integrable. Let $x_i : [t_0, t_1] \to \mathbb{R}^d$, $i = 1, 2$, be continuous weak solutions to*

$$x_i'(t) = \mathcal{K}_i(t)x_i(t) + f_i(t)$$

*and set $C = \sup_{t\in[t_0,t_1]} |x_2(t)|$. Then*

$$\begin{aligned}|x_1(t) - x_2(t)| \leq &|x_1(t_0) - x_2(t_0)|e^{L(t-t_0)} \\ &+ \int_{t_0}^{t} \big(C|\mathcal{K}_1(s) - \mathcal{K}_2(s)| + |f_1(s) - f_2(s)|\big)e^{L(t-s)}\, ds.\end{aligned}$$

**Proposition 2.2.8.** *Let $F_i : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, 2$ be Lipschitz with constant $L$ and let $x_i : [t_0, t_1] \to \mathbb{R}^d$ be $C^1$ solutions to*

$$x_i'(t) = F_i(x_i(t)).$$

*Suppose also that $|F_1(x) - F_2(x)| \leq \delta$ for $x \in x_2([t_0, t_1])$. Then*

$$|x_1(t) - x_2(t)| \leq |x_1(t_0) - x_2(t_0)|e^{L(t-t_0)} + \frac{\delta}{L}\big(e^{L(t-t_0)} - 1\big).$$

Our ultimate goal in this analysis will be to understand the behavior of stable solutions to (2.1.17), (2.1.18) as $\varepsilon \to 0$. It is clear from (2.1.19) and (2.1.4) that first one needs to study $\mathcal{A}_{r,\varepsilon}$ and $\rho_{s,r,\varepsilon}$. Certainly $\mathcal{A}_{r,\varepsilon}(\rho)$ is a $C^\infty$ function of $(\rho, r, \varepsilon)$ for $\varepsilon > 0$. Upon reducing to a first order system in the usual way, Proposition 2.2.6 implies that $\mathcal{A}_{r,0}(\rho)$ and $\mathcal{A}'_{r,0}(\rho)$ are continuous functions of $(\rho, r)$ which restrict to be $C^\infty$ on each of $\{r \geq \rho\}$ and $\{r \leq \rho\}$. The same argument as in the proof of Proposition 2.2.5 shows that $\partial_r\mathcal{A}_{r,0}(\rho)$ is continuous across $\rho = r$, so that $\mathcal{A}_{r,0}(\rho)$ is jointly $C^1$ everywhere. Our ultimate interest is in $r$ near $\pi/4$, so fix a small $\eta > 0$ and set $\mathcal{I} = [\pi/4 - \eta, \pi/4 + \eta]$.

**Proposition 2.2.9.** *For $k = 0, 1$, $\partial_\rho^k \mathcal{A}_{r,\varepsilon}(\rho) \to \partial_\rho^k \mathcal{A}_{r,0}(\rho)$ uniformly on compact subsets of $(\rho, r) \in [0, \infty) \times \mathcal{I}$ as $\varepsilon \to 0$. For $k \geq 2$, $\partial_\rho^k \mathcal{A}_{r,\varepsilon}(\rho) \to \partial_\rho^k \mathcal{A}_{r,0}(\rho)$ uniformly on compact subsets of $([0, \infty) \times \mathcal{I}) \cap \{\rho \neq r\}$ as $\varepsilon \to 0$.*

*Proof.* Reduce (2.1.6) to a first order system in the usual way: set

$$x = \begin{pmatrix} \mathcal{A} \\ \mathcal{A}' \end{pmatrix} \qquad \mathcal{K}_{r,\varepsilon} = \begin{pmatrix} 0 & 1 \\ -K^{\|}_{r,\varepsilon} & 0 \end{pmatrix}$$

so that (2.1.6) becomes $x' = \mathcal{K}x$, $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The first sentence follows from Proposition 2.2.7 since $K^{\|}_{r,\varepsilon} - K^{\|}_{r,0} \to 0$ in $L^1_{loc}([0, \infty))$ uniformly in $r$.

The convergence for $k \geq 2$ on $\{\rho < r\}$ is clear since $\mathcal{A}_{r,\varepsilon}(\rho)$ is independent of $\varepsilon \geq 0$ on that set. Equation (2.1.7) implies that as $\varepsilon \to 0$, eventually $\mathcal{A}_{r,\varepsilon}$ has the form $\mathcal{A}_{r,\varepsilon}(\rho) = a_+ e^\rho + a_- e^{-\rho}$ on any compact subset of $\{\rho > r\}$. The convergence for $k = 0$ implies that $a_\pm(r, \varepsilon) \to a_\pm(r, 0)$. The result for $k \geq 2$ therefore follows upon differentiation in $\rho$. $\square$

We now turn to geodesics. To streamline the notation we will often write $\nu = (r, \varepsilon)$, $\nu_0 = (\pi/4, 0)$, $\mu = (s, r, \varepsilon)$ and $\mu_0 = (0, \pi/4, 0)$. For example, we write $g_\nu := g_{r,\varepsilon}$ or $\gamma_\mu := \gamma_{s,r,\varepsilon}$. Recall that for $s \geq 0$, $\gamma_\mu(t)$ denotes a unit speed geodesic for $g_\nu$ whose distance from the origin equals $s$, parametrized so that this minimum distance is achieved at $t = 0$, and $\rho_\mu(t) = \rho(\gamma_\mu(t))$. For $s > 0$, $\rho_\mu(t)$ is the solution of (2.1.9) with $\mathcal{A} = \mathcal{A}_\nu$ and initial conditions $\rho(0) = s$, $\rho'(0) = 0$, while $\rho_{0,r,\varepsilon}(t) = t$ solves the same equation but has initial conditions $\rho(0) = 0$, $\rho'(0) = 1$. Throughout we restrict attention to $\varepsilon$ small and $r \in \mathcal{I}$, say $(r, \varepsilon) \in \mathcal{I} \times [0, \varepsilon_0]$ for fixed small positive $\varepsilon_0$. Often we consider $s$ to be small, so we also fix a small $s_0 > 0$ and in these situations we will assume $s \in [0, s_0]$. Despite the apparent difference in the initial conditions, (2.1.12) shows that $\rho_\mu(t)$ is smooth (and independent of $r$, $\varepsilon$) for $(t, s) \in ([0, t_0] \times [0, s_0]) \setminus \{(0, 0)\}$ for appropriately chosen $t_0$ small, and Lipschitz continuous for $(t, s) \in [0, t_0] \times [0, s_0]$. The different description of the initial conditions and the discontinuity of the first derivatives of $\rho_\mu(t)$ at $(t, s) = (0, 0)$ are a reflection of the singularity of polar coordinates at the origin.

A first observation is that $\rho_\mu(t) \to \infty$ as $t \to \infty$ uniformly for $(s, r, \varepsilon) \in [0, \infty) \times \mathcal{I} \times [0, \varepsilon_0]$. In fact, (2.1.7) together with the continuity of $a_\pm$ in $(r, \varepsilon)$ established in Proposition 2.2.9 imply that there is $a > 0$ so that $\mathcal{A}'_\nu(\rho)/\mathcal{A}_\nu(\rho) \geq a$ for $(r, \varepsilon) \in \mathcal{I} \times [0, \varepsilon_0]$ and $\rho > 0$. It follows that $\rho_\mu(t) \geq a^{-1} \log(\cosh(at))$ by the comparison argument in (2.1.10), (2.1.11).

We will analyze (2.1.9) with $\mathcal{A} = \mathcal{A}_{r,\varepsilon}$ by incorporating $r$ as an initial value and rewriting as $x' = \mathcal{F}_\varepsilon(x)$ with

$$x = \begin{pmatrix} \rho \\ v \\ r \end{pmatrix}, \qquad \mathcal{F}_\varepsilon(x) = \begin{pmatrix} v \\ \frac{\mathcal{A}'_{r,\varepsilon}(\rho)}{\mathcal{A}_{r,\varepsilon}(\rho)}(1 - v^2) \\ 0 \end{pmatrix} \tag{2.2.2}$$

and with initial conditions

$$x(0) = \begin{pmatrix} s \\ 0 \\ r \end{pmatrix} \quad \text{if } s > 0, \qquad x(0) = \begin{pmatrix} 0 \\ 1 \\ r \end{pmatrix} \quad \text{if } s = 0.$$

Our starting point is the following.

**Lemma 2.2.10.** *$\rho_\mu(t)$ is a continuous function of $(t, s, r, \varepsilon) \in [0, \infty) \times [0, \infty) \times \mathcal{I} \times [0, \varepsilon_0]$. $\rho'_\mu(t)$ and $\rho''_\mu(t)$ restrict to continuous functions on $\big([0, \infty) \times [0, \infty) \setminus \{(0, 0)\}\big) \times \mathcal{I} \times [0, \varepsilon_0]$.*

*Proof.* We have already discussed the regularity near $t = s = 0$. It is clear that $\rho_{s,r,\varepsilon}(t)$ restricts to a $C^\infty$ function of $(t, s, r, \varepsilon) \in \big([0, \infty) \times [0, \infty) \setminus \{(0, 0)\}\big) \times \mathcal{I} \times (0, \varepsilon_0]$. Now $\mathcal{F}_0$ is a locally Lipschitz function of $x$, so $\partial_t^l \rho_{s,r,0}(t)$ is a continuous function of $(t, s, r) \in \big([0, \infty) \times [0, \infty) \setminus \{(0, 0)\}\big) \times \mathcal{I}$ for $0 \leq l \leq 2$. Proposition 2.2.9 implies that $\mathcal{A}'_{r,\varepsilon}(\rho)/\mathcal{A}_{r,\varepsilon}(\rho)$ converges to the corresponding function evaluated at $\varepsilon = 0$ uniformly on compact subsets of $(\rho, r) \in (0, \infty) \times \mathcal{I}$. The fact that $\partial_t^l \rho_{s,r,\varepsilon} \to \partial_t^l \rho_{s,r,0}$ uniformly on compact subsets of $(t, s, r) \in \big([0, \infty) \times [0, \infty) \setminus (0, 0)\big) \times \mathcal{I}$ for $l = 0,\ 1$ follows from Proposition 2.2.8. The convergence of $\rho''$ as $\varepsilon \to 0$ then follows from the differential equation (2.1.9). $\square$

We will need to know similar continuity properties of solutions of (2.1.17) and (2.1.18) in our analysis of $s$-derivatives of $\rho$ and in later arguments.

**Lemma 2.2.11.** *Let $\mathcal{X}_\mu$ be any one of $\mathcal{U}_\mu^{\|}$, $\mathcal{U}_\mu^{\perp}$, $\mathcal{V}_\mu^{\|}$, or $\mathcal{V}_\mu^{\perp}$. Then $\mathcal{X}_\mu(t)$ and $\mathcal{X}_\mu'(t)$ are continuous functions of $(t,s,r,\varepsilon) \in [0,\infty) \times [0,\infty) \times \mathcal{I} \times [0,\varepsilon_0]$.*

*Proof.* First note that for all $(r,\varepsilon) \in \mathcal{I} \times [0,\varepsilon_0]$, $\mathcal{X}_\mu(t) = \sin(t)$ or $\cos(t)$ for $(t,s)$ near $(0,0)$. Rewrite (2.1.17) as the first order system $x' = \mathcal{K}x$ where

$$x = \begin{pmatrix} \mathcal{X} \\ \mathcal{X}' \end{pmatrix} \qquad \mathcal{K} = \mathcal{K}_\mu(t) = \begin{pmatrix} 0 & 1 \\ -K_\nu^{\|}(\rho_\mu(t)) & 0 \end{pmatrix}, \tag{2.2.3}$$

and likewise for (2.1.18). The functions $K_\nu^{\|}(\rho_\mu(t))$ and $K_\mu(t)$ are $C^\infty$ for $(t,s,r,\varepsilon) \in [0,\infty) \times [0,\infty) \times \mathcal{I} \times (0,\varepsilon_0]$. So $\mathcal{X}_\mu(t)$ is also $C^\infty$ on this same set. The functions $K_{r,0}^{\|}(\rho_{s,r,0}(t))$ and $K_{s,r,0}(t)$ are piecewise $C^\infty$ in $(t,s,r)$ with a jump discontinuity across $t = \ell_r(s)$. So Proposition 2.2.6 implies that $\mathcal{X}_{s,r,0}(t)$ is also piecewise $C^\infty$ with a jump discontinuity in second derivatives across $t = \ell_r(s)$. Recall from Lemma 2.2.10 that $\rho_{s,r,\varepsilon}$ and $\rho'_{s,r,\varepsilon}$ are continuous in $\varepsilon$ at $\varepsilon = 0$. So $K_{r,\varepsilon}^{\|} \circ \rho_{s,r,\varepsilon} - K_{r,0}^{\|} \circ \rho_{s,r,0} \to 0$, $K_{s,r,\varepsilon} - K_{s,r,0} \to 0$ in $L^1_{loc}([0,\infty))$ locally uniformly in $(s,r)$. Thus Proposition 2.2.7 implies that $x_{s,r,\varepsilon}(t) \to x_{s,r,0}(t)$ uniformly on compact subsets of $[0,\infty) \times [0,\infty) \times \mathcal{I}$. $\square$

Next we analyze continuity of higher derivatives of $\rho_\mu$, including $s$-derivatives. It will suffice for our needs to restrict attention to $s$ small, say $s \in [0,s_0]$ for $s_0 > 0$ small and fixed (as above). Set $\mathcal{R} = ([0,\infty) \times [0,s_0]) \setminus \{(0,0)\}$. For $\varepsilon = 0$, the problem (2.2.2) falls into the framework of Proposition 2.2.4 with the surface $S$ given by $\rho = r$, so $T\begin{pmatrix} s \\ 0 \\ r \end{pmatrix} = \ell_r(s) = \arccos\left(\frac{\cos r}{\cos s}\right)$. Proposition 2.2.5 shows that $\rho_{s,r,0}(t)$ and $\rho'_{s,r,0}(t)$ are $C^1$ functions of $(t,s,r) \in \mathcal{R} \times \mathcal{I}$ and Proposition 2.2.4 implies that $\rho_{s,r,0}(t)$ restricts to a $C^\infty$ function of $(t,s,r)$ on each of $(\mathcal{R} \times \mathcal{I}) \cap \{0 \le t \le \ell_r(s)\}$ and $(\mathcal{R} \times \mathcal{I}) \cap \{t \ge \ell_r(s)\}$.

**Proposition 2.2.12.** *Let $k,l \ge 0$. If $k+l \le 2$, then as $\varepsilon \to 0$, $\partial_s^k \partial_t^l \rho_{s,r,\varepsilon}(t)$ converges to the corresponding function evaluated at $\varepsilon = 0$, uniformly on compact subsets of $\mathcal{R} \times \mathcal{I}$. If $k+l = 3$ and $k < 3$, then $\partial_s^k \partial_t^l \rho_{s,r,\varepsilon}(t)$ converges to the corresponding function evaluated at $\varepsilon = 0$ uniformly on compact subsets of $(\mathcal{R} \times \mathcal{I}) \setminus \{t = \ell_r(s)\}$.*

*Proof.* The convergence for $k = 0$, $0 \le l \le 2$ is a specialization of Lemma 2.2.10. The stated convergence of $\rho_\mu'''$ follows upon differentiating (2.1.9) with respect to $t$.

We claim that

$$\mathcal{A}_\nu(\rho_\mu(t))\partial_s\rho_\mu(t) = \sin(s)\mathcal{U}_\mu^{\|}(t) \tag{2.2.4}$$

on $\mathcal{R} \times \mathcal{I} \times [0, \varepsilon_0]$. To see this, one verifies directly via the chain rule and the differential equations satisfied by $\mathcal{A}$ and $\rho$ that $\mathcal{A}_\nu(\rho_\mu(t))\partial_s\rho_\mu(t)$ is a solution (weak solution if $\varepsilon = 0$) to (2.1.17). (For $\varepsilon = 0$, recall that $\rho_{s,r,0}(t)$ and $\rho'_{s,r,0}(t)$ are $C^1$ functions of $(t, s, r)$.) Now (2.2.4) is easily checked directly for $t$ near 0 and $s \in [0, s_0]$, where we have explicit formulas for all involved quantities. So the two sides are solutions of the same differential equation which agree for $t$ small; hence they are equal.

We use (2.2.4) to reduce the study of $\partial_s\rho_\mu(t)$ to the study of $\mathcal{U}_\mu^{\|}(t)$. As for the factor $\mathcal{A}_\nu(\rho_\mu(t))$, Proposition 2.2.9 and Lemma 2.2.10 imply that $\mathcal{A}_\nu(\rho_\mu(t)) \to \mathcal{A}_{r,0}(\rho_{s,r,0}(t))$ and $\Big(\mathcal{A}_\nu(\rho_\mu(t))\Big)' \to \Big(\mathcal{A}_{r,0}(\rho_{s,r,0}(t))\Big)'$ uniformly on compact subsets of $\mathcal{R}\times\mathcal{I}$. So we deduce from (2.2.4) and Lemma 2.2.11 that $\partial_s\rho_\mu(t) \to \partial_s\rho_{s,r,0}(t)$ and $\partial_s\rho'_\mu(t) \to \partial_s\rho'_{s,r,0}(t)$ uniformly on compact subsets of $\mathcal{R} \times \mathcal{I}$. The differential equation (2.1.17) implies that $\mathcal{U}_{r,s,\varepsilon}^{\|}{}'' \to \mathcal{U}_{r,s,0}^{\|}{}''$ uniformly on compact subsets of $\Big([0,\infty) \times [0, s_0] \times \mathcal{I}\Big) \setminus \{t = \ell_r(s)\}$. Since $\Big(\mathcal{A}_\nu(\rho_\mu(t))\Big)'' \to \Big(\mathcal{A}_{r,0}(\rho_{s,r,0}(t))\Big)''$ uniformly on compact subsets of $(\mathcal{R} \times \mathcal{I}) \setminus \{t = \ell_r(s)\}$, it follows also that $\partial_s\rho''_\mu(t) \to \partial_s\rho''_{s,r,0}(t)$ uniformly on compact subsets of $(\mathcal{R} \times \mathcal{I}) \setminus \{t = \ell_r(s)\}$.

It remains to analyze $\partial_s^2\rho_\mu(t)$ and $\partial_s^2\rho'_\mu(t)$, which we will do by differentiating (2.2.4) with respect to $s$. Begin by considering $\partial_s\mathcal{U}_\mu^{\|}$. The equation for $\mathcal{U}_\mu^{\|}$ reduces to a first order system as in (2.2.3) with $\mathcal{X} = \mathcal{U}_\mu^{\|}$. Define $y := \partial_s x - \frac{\partial_s\rho}{\rho'}\mathcal{K}x$ on $(0,\infty) \times [0, s_0] \times \mathcal{I} \times [0, \varepsilon_0]$. We claim first that when $\varepsilon > 0$, $y$ solves the equation

$$y' = \mathcal{K}y + f(t), \qquad \text{where} \quad f(t) = -\left(\frac{\partial_s\rho}{\rho'}\right)'\mathcal{K}x. \tag{2.2.5}$$

To see this, note that the chain rule implies $\rho'\partial_s\mathcal{K} = (\partial_s\rho)\mathcal{K}'$. Then (2.2.5) follows by direct calculation:

$$\begin{aligned} y' =& \partial_s x' - \left(\frac{\partial_s \rho}{\rho'}\right)' \mathcal{K}x - \frac{\partial_s \rho}{\rho'}\mathcal{K}'x - \frac{\partial_s \rho}{\rho'}\mathcal{K}x' \\ =& \partial_s(\mathcal{K}x) + f(t) - (\partial_s \mathcal{K})x - \frac{\partial_s \rho}{\rho'}\mathcal{K}^2 x \\ =& \mathcal{K}\partial_s x + f(t) - \frac{\partial_s \rho}{\rho'}\mathcal{K}^2 x = \mathcal{K}y + f(t). \end{aligned} \tag{2.2.6}$$

If $\varepsilon = 0$, the same calculation leads to the same conclusion, but with all derivatives interpreted in the sense of distributions in $(s,t)$ near $t = \ell_r(s)$. In particular, for $\varepsilon = 0$, $y$ is a weak solution of (2.2.5), so is continuous across $t = \ell_r(s)$. The value $y(t)$ for $t$ small is independent of $r$, $\varepsilon$. Application of Proposition 2.2.7 therefore shows that $y_{s,r,\varepsilon} \to y_{s,r,0}$ uniformly on compact subsets of $(0,\infty) \times [0,s_0] \times \mathcal{I}$. The first component of $y$ is $\partial_s \mathcal{U}_\mu^{\|} - \frac{\partial_s \rho}{\rho'}\mathcal{U}_\mu^{\|}{}'$. We know that $\mathcal{U}_\mu^{\|}{}' \to \mathcal{U}_{s,r,0}^{\|}{}'$ uniformly on compact subsets of $[0,\infty) \times [0,s_0] \times \mathcal{I}$ by Lemma 2.2.11. So $\partial_s \mathcal{U}_\mu^{\|} \to \partial_s \mathcal{U}_{s,r,0}^{\|}$ uniformly on compact subsets of $(0,\infty)\times[0,s_0]\times\mathcal{I}$; hence uniformly on compact subsets of $[0,\infty)\times[0,s_0]\times\mathcal{I}$. The second component of $y$ is $\partial_s \mathcal{U}_\mu^{\|}{}' + \frac{\partial_s \rho}{\rho'}(K^{\|} \circ \rho)\mathcal{U}_\mu^{\|}$. It follows that $\partial_s \mathcal{U}_\mu^{\|}{}' \to \partial_s \mathcal{U}_{s,r,0}^{\|}{}'$ uniformly on compact subsets of $\big((0,\infty)\times[0,s_0]\times\mathcal{I}\big) \setminus \{t = \ell_r(s)\}$; hence uniformly on compact subsets of $\big([0,\infty) \times [0,s_0] \times \mathcal{I}\big) \setminus \{t = \ell_r(s)\}$.

Since $\partial_s\big(\mathcal{A}_\nu(\rho_\mu(t))\big) = \mathcal{A}_\nu'(\rho_\mu(t))\partial_s \rho_\mu(t)$ converges uniformly on compact subsets of $\mathcal{R}\times\mathcal{I}$ to the corresponding expression evaluated at $\varepsilon = 0$, applying $\partial_s$ to (2.2.4) shows that $\partial_s^2 \rho$ converges uniformly on compact subsets of $\mathcal{R} \times \mathcal{I}$ as claimed. Finally, one verifies easily via the chain rule and what we have already established that $\partial_s \partial_t\big(\mathcal{A}_\nu(\rho_\mu(t))\big)$ converges uniformly on compact subsets of $\{t \neq \ell_r(s)\}$. So applying $\partial_s \partial_t$ to (2.2.4) shows that $\partial_s^2 \rho'$ does too. □

We remark that it is easily seen from the arguments above that when $k+l = 3$ and $k < 3$, even though $\partial_s^k \partial_t^l \rho_\mu(t)$ is not uniformly convergent near $\{t = \ell_r(s)\}$ as $\varepsilon \to 0$, it is uniformly bounded near this set.

Next consider behavior as $t \to \infty$.

**Lemma 2.2.13.** *For $t$ large, $\rho_\mu$ can be written in the form*

$$\rho_\mu(t) = t + F(e^{-t}, s, r, \varepsilon) \tag{2.2.7}$$

*for a function $F$ satisfying $F \in C^\infty([0,1] \times [0,s_0] \times \mathcal{I} \times (0,\varepsilon_0])$ and $F|_{\varepsilon=0} \in C^\infty([0,1] \times [0,s_0] \times \mathcal{I})$.*

*Proof.* It must be shown that the function $F$ defined by (2.2.7) has the stated regularity properties for $t$ large. The geodesic flow $\varphi_t : S^*M \to S^*M$ of an asymptotically hyperbolic metric $g$ was analyzed in [GGS+]. The proof of Lemma 2.7 of [GGS+] shows that if $g$ is smooth and non-trapping and $u$ is a defining function for infinity, then for $t \geq 0$ one can write $u(\pi(\varphi_t(z))) = e^{-t}E(e^{-t}, z)$ for a smooth positive function $E$ on $[0,1] \times S^*M$. Here $\pi : S^*M \to M$ is the projection. Note that under the change of variable $u = e^{-\rho}$, for $\mathcal{A}$ of the form (2.1.7) the metric $g$ becomes $g = g^{a_+,a_-} = u^{-2}\big(du^2 + (a_+ + a_- u^2)^2 \mathring{g}\big)$ in a neighborhood $\mathcal{U}$ of $u = 0$. First let $a_\pm$ be fixed and set $\widetilde{\mathcal{U}}_{a_+,a_-} := \{z \in S^*_{g^{a_+,a_-}}M : \pi(\varphi_t(z)) \in \mathcal{U} \text{ for all } t \geq 0\}$. It follows that $\rho(\pi(\varphi_t(z))) = -\log u(\pi(\varphi_t(z))) = t + P(e^{-t}, z)$ where $P = -\log E \in C^\infty([0,1] \times \widetilde{\mathcal{U}}_{a_+,a_-})$. To incorporate the parameters $a_\pm$, let $\mathbf{A}$ denote the set of $(a_+, a_-)$ which arise as $(r, \varepsilon)$ varies over $\mathcal{I} \times [0, \varepsilon_0]$, set $\mathcal{S} := \{(a_+, a_-, z) : (a_+, a_-) \in \mathbf{A}, z \in \widetilde{\mathcal{U}}_{a_+,a_-}\} \subset \mathbb{R}^2 \times T^*M$, and view $P$ as defined on $[0,1] \times \mathcal{S}$. The argument of the proof of Lemma 2.7 of [GGS+] carries over to this setting and establishes that $P$ is smooth on $[0,1] \times \mathcal{S}$.

Fix $T$ large; for $t > T$ we have

$$F(e^{-t}, s, r, \varepsilon) = P\big(e^{-(t-T)}, a_+(r,\varepsilon), a_-(r,\varepsilon), \varphi_T(z_s)\big) \tag{2.2.8}$$

where $z_s$ is the point (independent of $(r, \varepsilon)$) in $T^*M$ corresponding to the initial data for $\gamma_\mu$ and $\varphi$ denotes the geodesic flow of $g_{r,\varepsilon}$. Now $a_+$, $a_-$ are $C^\infty$ functions of $r$, $\varepsilon$ for $\varepsilon > 0$, and are $C^\infty$ functions of $r$ when $\varepsilon = 0$. Likewise, $\gamma_\mu(t)$ is $C^\infty$ in all variables for $\varepsilon > 0$, and Proposition 2.2.4 implies that $\gamma_{s,r,0}(t)$ is $C^\infty$ in $(s, r, t)$ for $t$ large. The conclusion follows. ☐

It is easily verified that for $\mathcal{A} = a_+ e^\rho + a_- e^{-\rho}$, one has

$$K^\perp(\rho) = -1 + e^{-2\rho} G(e^{-2\rho}, a_+, a_-) \tag{2.2.9}$$

with $G \in C^\infty([0,1] \times \mathbf{A})$, where, as in the proof of Lemma 2.2.13, $\mathbf{A}$ is the set of all $(a_+, a_-) \in \mathbb{R}^2$ which arise for $(r, \varepsilon) \in \mathcal{I} \times [0, \varepsilon_0]$. Substituting (2.2.7), (2.2.9) into (2.1.19) and recalling that $K_\nu^{\|}(\rho_\mu(t))$ is identically $-1$ for $t$ large show that for $t$ large,

$$K_{s,r,\varepsilon}(t) = -1 + e^{-t} H(e^{-t}, s, r, \varepsilon) \tag{2.2.10}$$

where

$$H(e^{-t}, s, r, \varepsilon) = \big(2\partial_v F - e^{-t}(\partial_v F)^2\big)\, e^{-2\rho_\mu(t)} G\big(e^{-2\rho_\mu(t)}, a_+(r,\varepsilon), a_-(r,\varepsilon)\big). \tag{2.2.11}$$

Here $v = e^{-t}$ is the first argument of $F$ and $\partial_v F$ is evaluated at $(e^{-t}, s, r, \varepsilon)$. The function $H$ clearly satisfies the same conditions that $F$ satisfied in Lemma 2.2.13: $H \in C^\infty([0,1] \times [0, s_0] \times \mathcal{I} \times (0, \varepsilon_0])$ and $H|_{\varepsilon=0} \in C^\infty([0,1] \times [0, s_0] \times \mathcal{I})$.

Problem 29, p. 104 of [CL55] shows that there is a unique solution $\mathcal{Y}_\mu^\perp(t)$ to (2.1.18) for $t$ large for which $\lim_{t\to\infty} e^t \mathcal{Y}(t) = 1$. Moreover, it is not hard to show that the reasoning in the outlined solution of the cited problem in [CL55] shows that $\mathcal{Y}_\mu^\perp(t)$ has the same regularity in the parameters as $K_{s,r,\varepsilon}$: $\mathcal{Y}_\mu^\perp \in C^\infty([T,\infty) \times [0, s_0] \times \mathcal{I} \times (0, \varepsilon_0])$ and $\mathcal{Y}_{s,r,0}^\perp \in C^\infty([T,\infty) \times [0, s_0] \times \mathcal{I})$ for some large $T$. For $\varepsilon > 0$, $\mathcal{Y}_\mu^\perp$ extends to $t \geq 0$ as a solution with $\mathcal{Y}_\mu^\perp \in C^\infty([0,\infty) \times [0, s_0] \times \mathcal{I} \times (0, \varepsilon_0])$. For $\varepsilon = 0$ we can apply Proposition 2.2.6 backwards in time with initial data at $t = T$ to conclude that $\mathcal{Y}_{s,r,0}^\perp$ extends to $t \geq 0$ as a weak solution of (2.1.18), which is $C^1$ and piecewise $C^\infty$ in $(t, s, r)$, with a jump in second derivatives across $t = \ell_r(s)$.

Since $K_\nu^{\|}(\rho_\mu(t))$ is identically $-1$ for $t$ large uniformly for $(s, r, \varepsilon) \in [0, s_0] \times \mathcal{I} \times [0, \varepsilon_0]$, there is a unique solution (weak solution if $\varepsilon = 0$) $\mathcal{Y}_\mu^{\|}(t)$ to (2.1.17) which equals $e^{-t}$ for $t$ large. This solution $\mathcal{Y}_\mu^{\|}$ extends backwards to $[0,\infty)$ with the same regularity properties as $\mathcal{Y}_\mu^\perp(t)$.

**Proposition 2.2.14.** *Let $0 \leq l \leq 1$, $0 \leq k \leq 2$ and let $\mathcal{Y}_\mu$ be either $\mathcal{Y}_\mu^{\|}$ or $\mathcal{Y}_\mu^\perp$. As $\varepsilon \to 0$, $\partial_s^k \partial_t^l \mathcal{Y}_{s,r,\varepsilon}(t) \to \partial_s^k \partial_t^l \mathcal{Y}_{s,r,0}(t)$ uniformly on compact subsets of $[0,\infty) \times [0, s_0] \times \mathcal{I}$ for $0 \leq k + l \leq 1$, and uniformly on compact subsets of $\big([0,\infty) \times [0, s_0] \times \mathcal{I}\big) \setminus \{t = \ell_r(s)\}$ for $2 \leq k + l \leq 3$.*

*Proof.* First we claim that for $0 \leq k \leq 2$, $0 \leq l \leq 1$, and for fixed large $T$, $\partial_s^k \partial_t^l \mathcal{Y}_{s,r,\varepsilon}(t) \to \partial_s^k \partial_t^l \mathcal{Y}_{s,r,0}(t)$ as $\varepsilon \to 0$ uniformly on $[T,\infty) \times [0, s_0] \times \mathcal{I}$. This is clear for $\mathcal{Y}_\mu^{\|}$ since $\mathcal{Y}_\mu^{\|}(t) = e^{-t}$

for $t$ large. For $\mathcal{Y}_\mu^\perp$ this follows from the same argument in [CL55] proving the existence of $\mathcal{Y}_\mu^\perp$ if we establish that the function $H$ in (2.2.10) satisfies that for $0 \le k \le 2$, $0 \le l \le 1$ and $t$ large, $\partial_s^k \partial_t^l \big(H(e^{-t}, s, r, \varepsilon)\big)$ is uniformly bounded and continuous in $\varepsilon$ up to $\varepsilon = 0$. Recall that $F$ is given by (2.2.8). Since the equation 2.1.9 for $\rho$ decouples in the equations for the geodesic flow for $g_{r,\varepsilon}$, it is not hard to see that the argument of Lemma 2.7 of [GGS$^+$] cited in the proof of Lemma 2.2.13 applies directly to the $\rho$ equation so that in (2.2.8), $\varphi_T(z_s)$ (which amounts to $(\rho, \rho', \theta, \theta')$), can be replaced by only $(\rho_\mu(T), \rho'_\mu(T))$ on the right hand side. Since $P$ and $G$ are smooth, the uniform boundedness and continuity in $\varepsilon$ of $\partial_s^k \partial_t^l \big(H(e^{-t}, s, r, \varepsilon)\big)$ for $t$ large follow upon using (2.2.8) to express $F$ in terms of $P$ in (2.2.11), successively differentiating (2.2.11), applying the chain rule, and recalling Proposition 2.2.12.

Now we use the same sort of argument as in Proposition 2.2.12, but backwards in time. We write the rest of the proof for $\mathcal{Y}_\mu = \mathcal{Y}_\mu^\perp$; the argument for $\mathcal{Y}_\mu^\parallel$ is similar. Reduce (2.1.18) to a first order system $x' = \mathcal{K}x$, where

$$x = \begin{pmatrix} \mathcal{Y} \\ \mathcal{Y}' \end{pmatrix} \qquad \mathcal{K}_{s,r,\varepsilon}(t) = \begin{pmatrix} 0 & 1 \\ -K_{s,r,\varepsilon}(t) & 0 \end{pmatrix}$$

with $K_{s,r,\varepsilon}$ defined by (2.1.19). Our previous results imply that $K_{s,r,\varepsilon} \to K_{s,r,0}$ in $L^1([0,T])$, so Proposition 2.2.7 applied backwards in time with initial condition at $t = T$ shows that $x_{s,r,\varepsilon}(t) \to x_{s,r,0}(t)$ uniformly on $[0,T] \times [0, s_0] \times \mathcal{I}$. So the convergence also holds uniformly on $[0,\infty) \times [0, s_0] \times \mathcal{I}$. This proves the result for $k = 0$, $0 \le l \le 1$.

Define $y := \partial_s x - \frac{\partial_s \rho}{\rho'} \mathcal{K}x$ as in the proof of Proposition 2.2.12. This time the chain rule gives

$$\frac{\partial_s \rho}{\rho'} \mathcal{K}' = \partial_s \mathcal{K} + \kappa_{s,r,\varepsilon}(t) A, \tag{2.2.12}$$

where

$$\kappa = 2\big(\rho' \partial_s \rho' - (\partial_s \rho)\rho''\big)\big(K^\parallel \circ \rho - K^\perp \circ \rho\big), \qquad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

So the calculation analogous to (2.2.6) via the chain rule shows that

$$y' = \mathcal{K}y + f(t) \tag{2.2.13}$$

with

$$f = -\left(\frac{\partial_s \rho}{\rho'}\right)' \mathcal{K}x - \kappa A x, \tag{2.2.14}$$

and again the equation holds weakly across $t = \ell_r(s)$ when $\varepsilon = 0$. Since $\mathcal{K}_{s,r,\varepsilon} - \mathcal{K}_{s,r,0} \to 0$ and $f_{s,r,\varepsilon} - f_{s,r,0} \to 0$ in $L^1_{loc}((0,\infty))$ as $\varepsilon \to 0$, Proposition 2.2.7 implies that $y_{s,r,\varepsilon} \to y_{s,r,0}$ uniformly on compact subsets of $(0,\infty) \times [0, s_0] \times \mathcal{I}$. Consideration of the first component shows that $\partial_s \mathcal{Y}_{s,r,\varepsilon}(t) \to \partial_s \mathcal{Y}_{s,r,0}(t)$ uniformly on compact subsets of $(0,\infty) \times [0, s_0] \times \mathcal{I}$ and consideration of the second component shows that $\partial_s \mathcal{Y}'_{s,r,\varepsilon}(t) \to \partial_s \mathcal{Y}'_{s,r,0}(t)$ uniformly on compact subsets of $\big((0,\infty) \times [0, s_0] \times \mathcal{I}\big) \setminus \{t = \ell_r(s)\}$. Since $K_\mu(t) = 1$ for $t$ small uniformly in $(s, r, \varepsilon)$, the differential equation (2.1.18) implies that the uniform convergence extends down to $t = 0$. This proves the result for $k = 1$, $0 \le l \le 1$.

For $k = 2$, set

$$z = \partial_s y - \frac{\partial_s \rho}{\rho'}\mathcal{K}y + \left(\frac{\partial_s \rho}{\rho'}\right)' \frac{\partial_s \rho}{\rho'}\mathcal{K}x + \frac{\partial_s \rho}{\rho'}\kappa A x. \tag{2.2.15}$$

We claim that $z' = \mathcal{K}z + h(t)$, where $h(t) = h_{s,r,\varepsilon}(t)$ is given by

$$\begin{aligned} h(t) = &-\left(\frac{\partial_s \rho}{\rho'}\right)' \frac{\partial_s \rho}{\rho'}\mathcal{K}^2 x - \frac{\partial_s \rho}{\rho'}\kappa B x + \mathcal{K}\left[\left(\frac{\partial_s \rho}{\rho'}\right)' \frac{\partial_s \rho}{\rho'} x\right]' - \left(\frac{\partial_s \rho}{\rho'}\right)' \mathcal{K}y \\ &- \kappa A y - \frac{\partial_s \rho}{\rho'}\mathcal{K}f - \partial^2_{st}\left(\frac{\partial_s \rho}{\rho'}\right)\mathcal{K}x - \left(\frac{\partial_s \rho}{\rho'}\right)' \mathcal{K}\partial_s x - \kappa A \partial_s x \\ &- 2\Big[\big(\partial_s - \frac{\partial_s \rho}{\rho'}\partial_t\big)\big(\rho' \partial_s \rho' - (\partial_s \rho)\rho''\big)\Big](K^{\parallel} \circ \rho - K^{\perp} \circ \rho) A x \\ &+ 2\left(\frac{\partial_s \rho}{\rho'}\right)' \kappa A x + \frac{\partial_s \rho}{\rho'}\kappa A x' \end{aligned}$$

and we have set $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Given the claim, the proof is concluded by the same sort of reasoning as above. Note that our previous results imply that $h_{s,r,\varepsilon} \to h_{s,r,0}$ uniformly on compact subsets of $\big((0,\infty) \times [0, s_0] \times \mathcal{I}\big) \setminus \{t = \ell_r(s)\}$, and $h_{s,r,\varepsilon}$ is uniformly bounded on compact subsets of $(0,\infty) \times [0, s_0] \times \mathcal{I}$. So $\mathcal{K}$ and $h$ converge in $L^1_{loc}((0,\infty))$. Thus Proposition 2.2.7 shows that $z_{s,r,\varepsilon} \to z_{s,r,0}$ uniformly on compact subsets of $(0,\infty) \times [0, s_0] \times \mathcal{I}$. According to (2.2.15), $z - \partial_s y$ is the sum of three terms, each of which converges uniformly on compact subsets of $\big((0,\infty) \times [0, s_0] \times \mathcal{I}\big) \setminus \{t = \ell_r(s)\}$. So $\partial_s y$ also converges uniformly on compact subsets of $\big((0,\infty)\times[0,s_0]\times\mathcal{I}\big)\setminus\{t=\ell_r(s)\}$. And $\partial_s y-\partial_s^2 x=-\partial_s\Big(\frac{\partial_s\rho}{\rho'}\mathcal{K}x\Big)$ converges uniformly on compact subsets of $\big((0,\infty)\times[0,s_0]\times\mathcal{I}\big)\setminus\{t=\ell_r(s)\}$, so $\partial_s^2 x$ does too. Again the differential equation (2.1.18) implies that the uniform convergence extends to $t=0$.

The proof that $z'=\mathcal{K}z+h(t)$ is a calculation similar to (2.2.6), (2.2.13) but involving more terms. Differentiate (2.2.15) with respect to $t$, expand the differentiations using the Leibnitz rule, substitute (2.2.13) for the two occurrences of $y'$ and (2.2.12) for the two occurrences of $\frac{\partial_s\rho}{\rho'}\mathcal{K}'$ on the right-hand side, and collect terms. One obtains

$$\begin{aligned}
z' =&\mathcal{K}\left(\partial_s y-\frac{\partial_s\rho}{\rho'}\mathcal{K}y\right)+\partial_s f-\left(\frac{\partial_s\rho}{\rho'}\right)'\mathcal{K}y-\kappa Ay-\frac{\partial_s\rho}{\rho'}\mathcal{K}f\\
&+\left(\frac{\partial_s\rho}{\rho'}\right)'(\partial_s\mathcal{K})x+2\left(\frac{\partial_s\rho}{\rho'}\right)'\kappa Ax+\mathcal{K}\left[\left(\frac{\partial_s\rho}{\rho'}\right)'\frac{\partial_s\rho}{\rho'}x\right]'\\
&+\frac{\partial_s\rho}{\rho'}\kappa' Ax+\frac{\partial_s\rho}{\rho'}\kappa Ax'.
\end{aligned}$$

Now substitute

$$\partial_s y-\frac{\partial_s\rho}{\rho'}\mathcal{K}y=z-\left(\frac{\partial_s\rho}{\rho'}\right)'\frac{\partial_s\rho}{\rho'}\mathcal{K}x-\frac{\partial_s\rho}{\rho'}\kappa Ax$$

from (2.2.15) in the first term on the right-hand side, expand $\partial_s f$ by differentiating (2.2.14), and compare terms to obtain

$$\begin{aligned}
&z'-\mathcal{K}z-h(t)\\
&=\left[\left(\frac{\partial_s\rho}{\rho'}\kappa'-\partial_s\kappa\right)+2\Big[\big(\partial_s-\frac{\partial_s\rho}{\rho'}\partial_t\big)\big(\rho'\partial_s\rho'-(\partial_s\rho)\rho''\big)\Big](K^{\|}-K^{\perp})\circ\rho\right]Ax.
\end{aligned}$$

Finally, observe that the right-hand side vanishes. $\square$

It is easily checked that for $\mu_0=(0,\pi/4,0)$ the decaying solution $\mathcal{Y}^{\|}_{\mu_0}=\mathcal{Y}^{\perp}_{\mu_0}=:\mathcal{Y}_{\mu_0}$ is given by $\mathcal{Y}_{0,\pi/4,0}(t)=\begin{cases}\sqrt{2}e^{-\pi/4}\cos(t), & 0\le t\le\pi/4\\ e^{-t}, & t\ge\pi/4\end{cases}$. Since $\mathcal{Y}_{\mu_0}(0)>0$, it follows from continuity that $\mathcal{Y}^{\|}_{\mu}(0)>0$, $\mathcal{Y}^{\perp}_{\mu}(0)>0$ for all $\mu$ sufficiently close to $\mu_0$. For such $\mu$ we define $\mathcal{W}^{\|}_{\mu}(t)=\mathcal{Y}^{\|}_{\mu}(t)/\mathcal{Y}^{\|}_{\mu}(0)$ and $\mathcal{W}^{\perp}_{\mu}(t)=\mathcal{Y}^{\perp}_{\mu}(t)/\mathcal{Y}^{\perp}_{\mu}(0)$ so that $\mathcal{W}^{\|}_{\mu}$, $\mathcal{W}^{\perp}_{\mu}$ are the decaying solutions satisfying $\mathcal{W}_\mu(0) = 1$. These $\mathcal{W}_\mu$ inherit the continuity properties of $\mathcal{Y}_\mu$ stated in Proposition 2.2.14.

In the sequel we will need the following lemma.

**Lemma 2.2.15.** *There exists $T > 0$ so that $K_\nu^{\|}(\rho_\mu(t)) < 0$ and $K_\mu(t) < 0$ for $(t, s, r, \varepsilon) \in [T, \infty) \times [0, \infty) \times \mathcal{I} \times [0, \varepsilon_0]$.*

*Proof.* Since $\rho_\mu(t) \to \infty$ as $t \to \infty$ uniformly for $\mu = (s, r, \varepsilon) \in [0, \infty) \times \mathcal{I} \times [0, \varepsilon_0]$ and $K_\mu(t)$ is a convex combination of $K_\nu^{\|}(\rho_\mu(t))$ and $K_\nu^{\perp}(\rho_\mu(t))$, it suffices to show that there exists $\rho_0$ independent of $\nu = (r, \varepsilon) \in \mathcal{I} \times [0, \varepsilon_0]$ so that $K_\nu^{\|}(\rho) < 0$ and $K_\nu^{\perp}(\rho) < 0$ for $\rho \geq \rho_0$. For $K_\nu^{\|}$ this is clear since $K_\nu^{\|}(\rho) = -1$ for $\rho > r + \varepsilon$. Equation (2.1.7) and the continuity of $a_\pm$ in $(r, \varepsilon)$ show that we can choose $\rho_0$ independent of $\nu \in \mathcal{I} \times [0, \varepsilon_0]$ so that $\mathcal{A}_\nu'(\rho_0) > 1$. The differential equation for $\mathcal{A}_\nu$ implies that $\mathcal{A}_\nu'(\rho) \geq \mathcal{A}_\nu'(\rho_0) > 1$ for $\rho \geq \rho_0$. Then $K_\nu^{\perp}(\rho) = \mathcal{A}_\nu^{-2}(\rho)\big(1 - (\mathcal{A}_\nu'(\rho))^2\big) < 0$ as desired. □

**Lemma 2.2.16.** *$\mathcal{W}_\mu^{\|}(t) > 0$ and $\mathcal{W}_\mu^{\perp}(t) > 0$ for all $t \geq 0$ and for all $\mu$ sufficiently near $\mu_0$.*

*Proof.* We suppress the superscripts $^{\|}$, $^{\perp}$; the argument is the same for both. Recall the solution $\mathcal{V}_\mu$ with initial conditions $\mathcal{V}_\mu(0) = 0$, $\mathcal{V}_\mu'(0) = 1$. The Wronskian $\mathcal{W}_\mu \mathcal{V}_\mu' - \mathcal{W}_\mu' \mathcal{V}_\mu = 1$. We will show below that $\mathcal{V}_\mu(t) > 0$ for all $t > 0$. Given this, it follows that $\mathcal{W}_\mu'(t) < 0$ at every $t$ for which $\mathcal{W}_\mu(t) = 0$. The vanishing of $\mathcal{W}_\mu(t)$ for any $t$ is therefore inconsistent with the fact that $\mathcal{W}_\mu$ is asymptotic to a positive multiple of $e^{-t}$ as $t \to \infty$.

Now we show that $\mathcal{V}_\mu(t) > 0$ for all $t > 0$ for $\mu$ sufficiently close to $\mu_0$. $\mathcal{V}_{\mu_0}$ is identified in (2.1.21) (take $\ell(s) = \pi/4$) and clearly is positive on $(0, \infty)$. Choose $T$ as in Lemma 2.2.15. Continuity (from Lemma 2.2.11) and the fact that $\mathcal{V}_\mu(t) = \sin(t)$ for $t$ small imply that there is a neighborhood of $\mu_0$ for which $\mathcal{V}_\mu > 0$ and $\mathcal{V}_\mu' > 0$ on $(0, T]$. The differential equation (2.1.17) or (2.1.18) implies that $\mathcal{V}_\mu > 0$ on $[T, \infty)$ as desired. □

**Proposition 2.2.17.** *Let $\mathcal{W}_\mu(t)$ be either $\mathcal{W}_\mu^{\|}(t)$ or $\mathcal{W}_\mu^{\perp}(t)$. Then $\partial_s(\mathcal{W}_\mu'(0))\big|_{s=0} = 0$ and there exist a neighborhood $U$ of $(\pi/4, 0)$ and $\sigma > 0$ such that if $(r, \varepsilon) \in U$ and $0 \leq s \leq \sigma$, then $\partial_s^2(\mathcal{W}_\mu'(0)) < 0$.*

*Proof.* For the first statement we actually show $\partial_s(\mathcal{W}_\mu(t))\big|_{s=0} = 0$ for all $t$. It is clear from (2.2.4) that $\partial_s\rho\big|_{s=0} = \partial_s\rho'\big|_{s=0} = 0$. In case $\mathcal{W}_\mu = \mathcal{W}_\mu^\perp$, if $\varepsilon > 0$ differentiation of (2.1.18) shows that $\partial_s\mathcal{W}_\mu|_{s=0}$ is also a solution of (2.1.18). This holds in the weak sense when $\varepsilon = 0$ by the reasoning in the proof of Proposition 2.2.14. Since $\partial_s\mathcal{W}_\mu|_{s=0}$ vanishes as $t \to \infty$, it must be a multiple of $\mathcal{W}_\mu$. Since $\partial_s\mathcal{W}_\mu(0)|_{s=0} = 0$ and $\mathcal{W}_\mu(0) = 1$, the multiple must be zero. The same argument applies to $\mathcal{W}_\mu = \mathcal{W}_\mu^\parallel$ upon differentiation of (2.1.17).

For the second statement, it suffices to show that $\partial_s^2(\mathcal{W}'_{s,\pi/4,0}(0))|_{s=0} < 0$ by Proposition 2.2.14. Again consider first $\mathcal{W} = \mathcal{W}^\perp$ and suppress writing $^\perp$ on all quantities below. For $s$ small we can write $\mathcal{W}_s := \mathcal{W}_{s,\pi/4,0}$ as a linear combination of the solutions $\mathcal{U}_s$ and $\mathcal{V}_s$ given by (2.1.24), (2.1.25). By first considering the asymptotics as $t \to \infty$ and then the value at $t = 0$, one finds that for $s > 0$ small

$$\mathcal{W}_s = \mathcal{U}_s - \csc(s)\cot(\Theta_\infty(s))\mathcal{V}_s$$

where $\Theta_\infty$ is given by (2.1.26). Hence $\mathcal{W}_s'(0) = -\csc(s)\cot(\Theta_\infty(s))$. Evaluation of (2.1.26) gives $\Theta_\infty(0) = \pi/2$ and (2.1.27) shows that $\partial_s\Theta_\infty = -s^2/2 + O(s^3)$. Thus $\Theta_\infty(s) = \pi/2 - s^3/6 + O(s^4)$ so that $\cot(\Theta_\infty(s)) = s^3/6 + O(s^4)$. This gives $\partial_s^2\mathcal{W}_s'(0)|_{s=0} = -1/3$ as desired. For the second case $\mathcal{W} = \mathcal{W}^\parallel$, write $\mathcal{W}_s$ as a linear combination of the solutions (2.1.20) and (2.1.21) and find, also using (2.1.13),

$$\mathcal{W}_s = \mathcal{U}_s - \frac{1-\sqrt{\cos(2s)}}{1+\sqrt{\cos(2s)}}\mathcal{V}_s,$$

so that $\mathcal{W}_s'(0) = -\frac{1-\sqrt{\cos(2s)}}{1+\sqrt{\cos(2s)}}$. This time there are no indeterminants and one finds without difficulty $\partial_s^2\mathcal{W}_s'(0)|_{s=0} = -1$. □

We will use the next proposition to rule out interior conjugate points for $s$ near 0.

**Proposition 2.2.18.** *Let $f \in L^1(\mathbb{R})$ be an even function and suppose $\mathcal{W}$ is a $C^1$ weak solution to*

$$\mathcal{W}''(t) + (-1 + f(t))\mathcal{W}(t) = 0 \tag{2.2.16}$$

*with $\mathcal{W}(t) > 0$ for $t \geq 0$ and $\lim_{t\to\infty} \mathcal{W}(t) = 0$. There are no nontrivial solutions of* (2.2.16) *vanishing at two distinct values of $t$ if and only if $\mathcal{W}'(0) \leq 0$.*

*Proof.* First assume that $\mathcal{W}'(0) \leq 0$. By the Sturm Separation Theorem, all solutions of (2.2.16) vanish at most once if there exists one solution of (2.2.16) that never vanishes, so it is enough to show that $\mathcal{W}(t) \neq 0$ for all $t$. If $\mathcal{W}'(0) = 0$ then $\mathcal{W}(|t|)$ is a non-vanishing $C^1$ solution of (2.2.16), so suppose $\mathcal{W}'(0) < 0$ and let $t_0 < 0$ be such that $\mathcal{W}(t_0) = 0$ for the sake of contradiction. Define $\mathcal{V} : \mathbb{R} \to \mathbb{R}$ by $\mathcal{V}(t) = \mathcal{W}(t) - \mathcal{W}(-t)$. Then $\mathcal{V}$ satisfies (2.2.16) by evenness of $f$. Since $\mathcal{V}(-t_0) > 0$, $\mathcal{V}(0) = 0$ and $\mathcal{V}'(0) < 0$, there exists $0 < t_1 < -t_0$ with $\mathcal{V}(t_1) = 0$. This contradicts the Sturm Separation Theorem, since $\mathcal{W}$ and $\mathcal{V}$ are linearly independent and $\mathcal{W} > 0$ on $[0, -t_1]$.

For the converse, suppose that no solutions of (2.2.16) vanish twice and $\mathcal{W}'(0) > 0$. We can normalize to assume $\mathcal{W}(0) = 1$. Let $\mathcal{U}$ denote the solution of (2.2.16) with $\mathcal{U}(0) = 1$ and $\mathcal{U}'(0) = 0$, which is even by evenness of $f$. We claim that $\mathcal{U}$ vanishes for some positive $t$, hence twice. Suppose this is not the case, i.e. $\mathcal{U} > 0$ on $\mathbb{R}$. We have $0 < \mathcal{U}(t) < \mathcal{W}(t)$ for all $t > 0$; otherwise the function $\mathcal{W} - \mathcal{U}$ would vanish at least twice on $[0, \infty)$. We conclude that $\lim_{t\to\infty} \mathcal{U}(t) = 0$ and hence all solutions of (2.2.16) decay as $t \to \infty$. This is a contradiction: Problem 29 in p. 104 of [CL55] implies that there are solutions of (2.2.16) which grow exponentially as $t \to \infty$. $\square$

Finally we can prove Propositions 2.2.1, 2.2.2 and 2.2.3.

*Proof of Proposition 2.2.1.* It is easy to check that for $g_{r,0}$ the decaying solution on radial geodesics is given for $t \geq 0$ by

$$\mathcal{Y}_{0,r,0}(t) = \begin{cases} e^{-r}\big(\cos(t-r) - \sin(t-r)\big) & t \leq r \\ e^{-t} & t \geq r \end{cases}.$$

(Since $\mathcal{Y}^{\parallel}_{0,r,\varepsilon} = \mathcal{Y}^{\perp}_{0,r,\varepsilon}$, we suppress the $^{\parallel}$.) So $\mathcal{Y}'_{0,r,0}(0) = e^{-r}\big(\sin(r) - \cos(r)\big)$. If $r_1 < \pi/4 < r_2$, then $\mathcal{Y}'_{0,r_1,0}(0) < 0 < \mathcal{Y}'_{0,r_2,0}(0)$ and we can choose $r_1$ and $r_2$ as close to $\pi/4$ as we like. Continuity (from Proposition 2.2.14) implies that if $\varepsilon$ is small enough, then also

$\mathcal{Y}'_{0,r_1,\varepsilon}(0) < 0 < \mathcal{Y}'_{0,r_2,\varepsilon}(0)$. The mean value theorem gives the existence of $r$, $r_1 < r < r_2$, with $\mathcal{Y}'_{0,r,\varepsilon}(0) = 0$. $\square$

*Proof of Proposition 2.2.2.* Choose $\sigma$ and $U$ so that the conclusion $\partial_s^2(\mathcal{W}'_\mu(0)) < 0$ for $(r,\varepsilon) \in U$ and $0 \leq s \leq \sigma$ of Proposition 2.2.17 holds for both $\mathcal{W}^{\|}_\mu$ and $\mathcal{W}^{\perp}_\mu$. The hypothesis ${\mathcal{Y}^{\|}_{0,r,\varepsilon}}'(0) = 0$ certainly implies that ${\mathcal{W}^{\|}_{0,r,\varepsilon}}'(0) = 0$, and also we have ${\mathcal{W}^{\perp}_{0,r,\varepsilon}}'(0) = 0$ since $\mathcal{W}^{\|}_{0,r,\varepsilon} = \mathcal{W}^{\perp}_{0,r,\varepsilon}$. Combining this with $\partial_s\big(\mathcal{W}'_{s,r,\varepsilon}(0)\big)\big|_{s=0} = 0$, it follows that $\mathcal{W}'_{s,r,\varepsilon}(0) \leq 0$ for $0 \leq s \leq \sigma$ for both $\mathcal{W}^{\|}_\mu$ and $\mathcal{W}^{\perp}_\mu$. Proposition 2.2.18 then implies that along any geodesic $\gamma_\mu \subset \Sigma_\gamma$ with $0 \leq s \leq \sigma$, no nontrivial normal Jacobi field which is either tangent to $\Sigma_\gamma$ or normal to $\Sigma_\gamma$ can vanish twice. Proposition 2.1.4 shows that no nontrivial normal Jacobi field can vanish twice, just as in the proof for $g_{\pi/4,0}$. Hence $g_{r,\varepsilon}$ has no interior conjugate points on a geodesic $\gamma_\mu$ for which $0 \leq s \leq \sigma$. $\square$

*Proof of Proposition 2.2.3.* First we claim that there exists $S > 0$ so that for any $(r,\varepsilon) \in \mathcal{I} \times [0,\varepsilon_0]$, $g_{r,\varepsilon}$ has no interior conjugate points on any geodesic $\gamma_{s,r,\varepsilon}$ with $s \geq S$. To see this, recall that we showed in the proof of Lemma 2.2.15 that there is $\rho_0 > 0$ independent of $\nu \in \mathcal{I} \times [0,\varepsilon_0]$ so that $K^{\|}_\nu(\rho) < 0$ and $K^{\perp}_\nu(\rho) < 0$ for $\rho \geq \rho_0$. Since for any $(r,\varepsilon)$, $s = \min_{t\in\mathbb{R}} \rho_{s,r,\varepsilon}(t)$, we know that if $s \geq \rho_0$, then $\rho_\mu(t) \geq \rho_0$ for all $t \in \mathbb{R}$. It follows that $K^{\|}_\nu(\rho_\mu(t)) < 0$ and $K_\mu(t) < 0$ for $t \in \mathbb{R}$ so long as $s \geq \rho_0$ and $(r,\varepsilon) \in \mathcal{I} \times [0,\varepsilon_0]$. Since the equation $Y'' = 0$ has a nonvanishing solution on $\mathbb{R}$, the Sturm Comparison Theorem implies that if $s \geq \rho_0$ and $(r,\varepsilon) \in \mathcal{I} \times [0,\varepsilon_0]$, then no nontrivial solution of (2.1.17) or (2.1.18) can vanish twice. This gives the claim with $S = \rho_0$ upon recalling Proposition 2.1.4.

We will now show that given any $\sigma > 0$, there is a neighborhood $V$ of $(\pi/4, 0)$ such that $\mathcal{U}^{\|}_\mu(t)$ and $\mathcal{U}^{\perp}_\mu(t)$ are positive for all $t \in \mathbb{R}$ for $(r,\varepsilon) \in V$ and $\sigma \leq s \leq S$, thus excluding nontrivial solutions vanishing twice by the Sturm Separation Theorem. It suffices to consider $t \geq 0$ since $\mathcal{U}^{\|}_\mu(t)$ and $\mathcal{U}^{\perp}_\mu(t)$ are even. Choose $T$ as in Lemma 2.2.15. We showed in the proofs of Lemmas 2.1.6 and 2.1.7 that $\mathcal{U}^{\|}_{s,\pi/4,0}(t)$ and $\mathcal{U}^{\perp}_{s,\pi/4,0}(t)$ are everywhere positive for any $s \geq 0$, and that analysis also shows that these solutions grow exponentially as $t \to \infty$ uniformly for $s \in [\sigma, S]$. Increasing $T$ if necessary, continuity (from Lemma 2.2.11) implies that there is a neighborhood $V$ of $(\pi/4, 0)$ and $c > 0$ so that $\mathcal{U}_\mu^\|(t) \geq c$, $\mathcal{U}_\mu^\perp(t) \geq c$ for $0 \leq t \leq T$, $(r, \varepsilon) \in V$ and $0 \leq s \leq S$, and also $\mathcal{U}_\mu^{\|}{}'(T) > 0$, $\mathcal{U}_\mu^{\perp}{}'(T) > 0$ for $(r, \varepsilon) \in V$ and $\sigma \leq s \leq S$. The differential equations satisfied by $\mathcal{U}_\mu^\|$ and $\mathcal{U}_\mu^\perp$ then imply that the solutions stay positive for $t > T$. $\square$

# Chapter 3

# STABILITY ESTIMATES FOR THE X-RAY TRANSFORM ON SIMPLE AH MANIFOLDS

In this chapter we show Theorem 3, a stability estimate for the normal operator of the X-ray transform on a simple AH manifold, using the 0-pseudodifferential calculus of Mazzeo and Melrose ([MM87]). The chapter is organized as follows: in Section 3.1 we provide some background on the geodesic flow and the X-ray transform on AH manifolds following [GGS$^+$]. Section 3.2 contains background material on the 0-geometry and 0-calculus that will be needed later. In Section 3.3 we prove a lemma related to the exponential map on a simple AH manifold and use it to analyze the distance function and show that the normal operator $\mathcal{N}_g$ is an elliptic pseudodifferential operator in the 0-calculus. In Section 3.4 we identify the model operator of $\mathcal{N}_g$, which is invertible by the work of Berenstein and Casadio Tarabusi [BC91]. Finally, in Section 3.5 we construct a parametrix for $\mathcal{N}_g$, use it to show boundary regularity for elements in its nullspace, and prove Theorem 3. Throughout the chapter we use Einstein notation, with Latin indices running from 0 to $n$ and Greek indices from 1 to $n$.

## 3.1 Geodesic Flow of AH Manifolds and the X-Ray Transform

As already discussed briefly in the Introduction, the behavior of the geodesic flow on an AH manifold is more complicated compared to the case of a compact manifold with boundary. The orbits of the geodesic flow in the cotangent bundle of a non-trapping compact manifold with boundary $X$ can be parametrized using the "incoming boundary", $\partial_- SX := \{(z,\xi) \in S^*X : z \in \partial X \text{ and } \xi(\nu) < 0\}$, where $\nu$ stands for the outward pointing normal. An analogous way of parametrizing the geodesic flow in the AH setting was formulated in [GGS$^+$] and we recall it in this section. Most of this section is based on material there.

Let $(\mathring{M}^{n+1}, g)$ be a non-trapping AH manifold, with $\mathring{M}$ being the interior of a compact manifold with boundary $M$ (note that the notation of this chapter differs from the previous ones). Recall from the Introduction that a conformal representative $h$ in the conformal infinity of $g$ determines a boundary defining function $x$ for $\partial M$, called *geodesic boundary defining function* associated to $h$, such that $x^2 g\big|_{T\partial M} = h$ and $|dx|_{x^2 g} = 1$ near $\partial M$. Then via the flow of its gradient, $x$ induces a product decomposition of a collar neighborhood of $\partial M$ as $[0, \varepsilon)_x \times \partial M$, in terms of which the metric is written in normal form near $\partial M$

$$g = \frac{dx^2 + h_x}{x^2}, \tag{3.1.1}$$

where $h_x$ is a smooth 1-parameter family of metrics on $\partial M$ satisfying $h_0 = h$. Choosing coordinates $y^\alpha$ for $\partial M$ near a boundary point we can write $g = \frac{dx^2 + (h_x)_{\alpha\beta} dy^\alpha dy^\beta}{x^2}$.

The space of geodesics on $\mathring{M}$ can be parametrized by introducing an appropriate extension of the unit cosphere bundle $S^*\mathring{M} = \{(z, \xi) \in T^*\mathring{M} : |\xi|_g = 1\}$ down to $\partial M$. Recall that Melrose's b-cotangent bundle (see [Mel93]) is a smooth bundle over $M$ with natural projection $\pi$, canonically isomorphic with $T^*\mathring{M}$ over $\mathring{M}$ and trivialized locally near the boundary by $(dx/x, dy^1, \ldots, dy^n)$. Thus via the identification $T^*\mathring{M} \leftrightarrow ({}^bT^*M)^\circ$, $S^*\mathring{M}$ can be viewed as a subset of $({}^bT^*M)^\circ$ given near $\partial M$ by $\{(z, \xi = \overline{\zeta}\frac{dx}{x} + \eta_\alpha dy^\alpha) \in ({}^bT^*M)^\circ : \overline{\zeta}^2 + x^2|\eta|^2_{h_x} = 1\}$. Hence the closure of $S^*\mathring{M}$ in ${}^bT^*M$ is a smooth embedded non-compact submanifold of ${}^bT^*M$ with disconnected boundary; we denote this submanifold by $S^*M$. Moreover, the Hamiltonian vector field $X$ on $S^*\mathring{M}$ associated with the metric Lagrangian $\mathcal{L}_g = |\xi|^2_g/2$ can be written as $X = x\overline{X}$, where $\overline{X}$ extends to be smooth on $S^*M$ and transversal to its boundary: in coordinates it takes the form

$$\overline{X} = \overline{\zeta}\partial_x + x h_x^{\alpha\beta}\eta_\alpha \partial_{y^\beta} - \big(x|\eta|^2_{h_x} + \frac{1}{2}x^2\partial_x|\eta|^2_{h_x}\big)\partial_{\overline{\zeta}} - \frac{1}{2}x\partial_{y^\alpha}|\eta|^2_{h_x}\partial_{\eta_\alpha}.$$

The flow of $\overline{X}$ is incomplete and, since $X$ and $\overline{X}$ are related by multiplication by a scalar function, their integrals curves in $S^*\mathring{M}$ agree up to reparametrization. Orbits of the flow of $\overline{X}$ can be parametrized by their "incoming" covector, that is, each orbit can be identified with its intersection with the connected component of $\partial S^*M$ on which $\overline{X}$ is inward pointing. This component, often referred to as the incoming boundary, will be denoted by $\partial_- S^*M$ and it can be written as $\partial_- S^*M = \{(z, \xi = \overline{\zeta}\frac{dx}{x} + \eta_\alpha dy^\alpha) \in {}^bT^*M : x = 0,\, \overline{\zeta} = 1\}$. The definition of the outcoming boundary $\partial_+ S^*M$ is analogous, except $\overline{\zeta} = -1$ there. Both of those sets are invariant subsets of ${}^bT^*M\big|_{\partial M}$, independent of the choice of coordinates and of $g$. Given a choice of conformal representative (which induces a geodesic boundary defining function), $\partial_\pm S^*M$ can be identified with $T^*\partial M$ via $\mp x^{-1}dx + \eta^\alpha dy^\alpha \leftrightarrow \eta_\alpha dy^\alpha$.

The unit cosphere bundle $S^*\mathring{M}$ has a natural measure $d\lambda$ called the Liouville measure, induced by the restriction to $S^*\mathring{M}$ of the $2n+1$ form $\lambda = \alpha \wedge (d\alpha)^n$, with $\alpha$ the tautological 1-form on $T^*\mathring{M}$. This measure decomposes as $d\lambda = dV_g d\mu_g$, where $d\mu_g$ is the measure induced by $g$ on each fiber of $S^*\mathring{M}$ and $dV_g$ is the Riemannian volume density on $\mathring{M}$. As shown in [GGS+] (Lemma 2.2), $xd\lambda$ extends from $S^*\mathring{M}$ to a smooth measure on $S^*M$. Moreover, $\iota_X\lambda$ extends to a smooth $2n$-form on $S^*M$, which restricts to a volume form on $\partial_- S^*M$; the latter agrees with the canonical volume form on $T^*\partial M$ (induced by the symplectic form there) under the identification described above. We will denote the corresponding measure on $\partial_- S^*M$ by $d\lambda_\partial$.

Now let $f \in C_c^\infty(S^*\mathring{M})$ and $\varphi_t$ be the flow of the Hamiltonian vector field $X$ on $S^*\mathring{M}$, which is complete. We define the X-ray transform

$$If(z,\xi) = \int_{-\infty}^{\infty} f(\varphi_t(z,\xi))dt \in C_X^\infty(S^*\mathring{M}), \tag{3.1.2}$$

where the space $C_X^\infty(S^*\mathring{M})$ consists of smooth functions on $S^*\mathring{M}$ constant along the orbits of $X$. Since $C_c^\infty(\mathring{M})$ can be naturally viewed as a subset of $C_c^\infty(S^*\mathring{M})$ via pullback, (3.1.2) reduces to the usual X-ray transform on $C_c^\infty(\mathring{M})$ viewed as an element of $C_X^\infty(S^*\mathring{M})$. Now as we mentioned before the vector field $\overline{X} = x^{-1}X$ extends to be smooth on $S^*M$ and transverse to $\partial S^*M$. This implies that any $u \in C_X^\infty(S^*\mathring{M})$ extends smoothly to $S^*M$ down to $\partial_\pm S^*M$: by transversality, the flow of $\overline{X}$ running forward and backward can be used to identify a neighborhood of any point in $\partial_\mp S^*M$ respectively with a subset of $[0,\varepsilon)_t \times \partial_\mp S^*M$; then in terms of this decomposition $u$ is independent of $t$ and thus extends smoothly down to $t = 0$.

Therefore the restriction $u\big|_{\partial_\pm S^*M} \in C^\infty(\partial_\pm S^*M)$ is well defined; conversely, any function in $C^\infty(\partial_\pm S^*M)$ can be extended off of $\partial_\pm S^*M$ to be constant along the orbits of $\overline{X}$ in $S^*M$, and hence also those of $X$ in $S^*\mathring{M}$, thus yielding an element of $C^\infty_X(S^*\mathring{M})$. This discussion implies that we have an isomorphism

$$C^\infty_X(S^*\mathring{M}) \to C^\infty(\partial_- S^*M) \tag{3.1.3}$$

and both spaces are also isomorphic to $C^\infty_{\overline{X}}(S^*M) = C^\infty(S^*M) \cap \ker \overline{X}$. Note that due to these facts (3.1.2) can also be regarded as an element of $C^\infty_{\overline{X}}(S^*M)$, and of $C^\infty(\partial_- S^*M)$ upon restricting. In fact, the discussion on short geodesics in [GGS$^+$, Section 2.2] indicates that the range of $I$ is smaller than $C^\infty_{\overline{X}}(S^*M)$ whenever acting on $C^\infty_c(S^*\mathring{M})$ (or $C^\infty_c(\mathring{M})$). Indeed, it follows from there that given any compact set $K \subset S\mathring{M}$ there exists a compact set $K' \subset \partial_- S^*M$ such that any integral curve of $\overline{X}$ starting at $(z,\xi) \notin K'$ does not intersect $K$. Moreover, given a compact $K' \subset \partial_- S^*M$, the union of all integral curves of $\overline{X}$ starting at $K'$ forms a compact subset of $S^*M$. This allows us to conclude that $I : C^\infty_c(S^*\mathring{M}) \to C^\infty_{c,\overline{X}}(S^*M)$, where $C^\infty_{c,\overline{X}}(S^*M) = C^\infty_c(S^*M) \cap \ker(\overline{X})$.

The X-ray transform can be expressed using the flow $\overline{\varphi}_\tau$ of $\overline{X}$. As already mentioned, $\overline{\varphi}_\tau$ is not complete and it is a reparametrization of the flow $\varphi_t$ of $X$ in $S^*\mathring{M}$: for $(z,\xi) \in S^*\mathring{M}$ one has $\overline{\varphi}_\tau(z,\xi) = \varphi_t(z,\xi)$ with $t(\tau,(z,\xi)) = \int_0^\tau \frac{d\sigma}{x \circ \overline{\varphi}_\sigma(z,\xi)}$. Moreover, for each $(z,\xi) \in S^*M$ there exist finite $\tau_\pm(z,\xi) \geq 0$ such that $\overline{\varphi}_{\pm\tau_\pm(z,\xi)}(z,\xi) \in \partial_\pm S^*M$. Thus for $f \in C^\infty_c(S^*\mathring{M})$ (or $f \in C^\infty_c(\mathring{M})$) (3.1.2) can be rewritten as

$$If(z,\xi) = \int_0^{\tau_+(z,\xi)} f(\overline{\varphi}_\tau(z,\xi)) \frac{d\tau}{x \circ \overline{\varphi}_\tau(z,\xi)} \in C^\infty_{c,\overline{X}}(S^*M).$$

One can identify a formal adjoint $I^*$ of $I$ on appropriate function spaces using suitably chosen inner products. By [GGS$^+$, Lemma 3.6], there is an analog of Santaló's formula:

$$\int_{S^*\mathring{M}} f\, d\lambda = \int_{\partial_- S^*M} If\, d\lambda_\partial, \quad f \in C^\infty_c(S^*\mathring{M}). \tag{3.1.4}$$

Note that this implies that $I$ also extends continuously as an operator $I : L^1(S^*\mathring{M}; d\lambda) \to L^1(\partial_- S^*M; d\lambda_\partial)$ (where the isomorphism (3.1.3) is used implicitly). We define an inner product on $C^\infty_{c,\overline{X}}(S^*M)$: for $u_1, u_2 \in C^\infty_{c,\overline{X}}(S^*M)$ let

$$\langle u_1, u_2\rangle_\partial := \int_{\partial_- S^*M} u_1\, \overline{u_2}\, d\lambda_\partial,$$

where on the right hand side $u_1$, $u_2$ are restricted to $\partial_- S^*M$; we will generally not write this restriction explicitly. Now consider the X-ray transform viewed as an operator $I : C^\infty_c(M) \to C^\infty_{c,\overline{X}}(S^*M)$ and define the operator $I^* : C^\infty_{c,\overline{X}}(S^*M) \to C^\infty(\mathring{M})$ by

$$I^*u(z) = \int_{S^*_z\mathring{M}} u\, d\mu_g, \quad u \in C^\infty_{c,\overline{X}}(S^*M).$$

Considering real valued functions $f \in C^\infty_c(\mathring{M})$ and $u \in C^\infty_{c,\overline{X}}(S^*M)$, we use (3.1.4) to compute

$$\begin{aligned}\langle u, If\rangle_\partial &= \int_{\partial_- S^*M} u\, If\, d\lambda_\partial = \int_{\partial_- S^*M} I(uf)\, d\lambda_\partial \\ &= \int_{S^*\mathring{M}} u\, f d\lambda = \int_{\mathring{M}} \Big(\int_{S^*_z\mathring{M}} u\, d\mu_g\Big) f(z) dV_g(z) = \langle I^*u, f\rangle_{L^2(M;dV_g)}.\end{aligned} \tag{3.1.5}$$

This computation implies that with the stated inner products and function spaces $I^*$ is a formal adjoint for $I$.

We will later need to consider the X-ray transform and the normal operator $\mathcal{N}_g = I^*I$ acting on functions that live in weighted $L^2$ spaces. The target space of $I$ will also have to be an appropriately weighted $L^2$ space and as will become apparent soon it is more natural for this discussion to view $If$ as a function on $\partial_- S^*M$. Restriction to $\partial_- S^*M$ induces an isometry between $C^\infty_{c,\overline{X}}(S^*M)$ and $C^\infty_c(\partial_- S^*M)$ with respect to the the inner product $\langle\cdot,\cdot\rangle_\partial$ and the $L^2(\partial_- S^*M; d\lambda_\partial)$ inner product respectively, so (3.1.5) can also be rewritten as

$$\langle u, If\rangle_{L^2(\partial_- S^*M;d\lambda_\partial)} = \langle I^*u, f\rangle_{L^2(M;dV_g)}, \quad f \in C^\infty_c(\mathring{M}), \quad u \in C^\infty_{c,\overline{X}}(S^*M). \tag{3.1.6}$$

By [GGS+, Lemma 3.8] $I$ extends to a bounded operator $I : |\log x|^{-\beta} L^2(S^*\mathring{M}; d\lambda) \to L^2(\partial_- S^*M; d\lambda_\partial)$ provided $\beta > 1/2$. This also implies that $I : |\log x|^{-\beta} L^2(M; dV_g) \to L^2(\partial_- S^*M; d\lambda_\partial)$ is bounded. Hence by (3.1.6) $I^*$ extends to a bounded operator $I^* : L^2(\partial_- S^*M; d\lambda_\partial) \to |\log x|^{\beta} L^2(M; dV_g)$ for $\beta > 1/2$ (where the action of $I^*$ on a function $u \in L^2(\partial_- S^*M; d\lambda_\partial)$ is understood as an action on the extension of $u$ to $S^*M$ so that it is constant along the orbits of $\overline{X}$). Thus (3.1.6) is valid for $u \in L^2(\partial_- S^*M; d\lambda_\partial)$ and $f \in |\log x|^{-\beta} L^2(M; dV_g)$. Moreover, the normal operator is bounded

$$\mathcal{N}_g = I^*I : |\log x|^{-\beta} L^2(\mathring{M}; dV_g) \to |\log x|^{\beta} L^2(\mathring{M}, dV_g), \quad \beta > 1/2.$$

Using the microlocal properties of $\mathcal{N}_g$ that we prove in Section 3.3 and Lemma 3.1.1 below, we will obtain extensions of $I$ and $I^*I$ to larger spaces of functions, in Corollary 3.3.5.

The following lemma relates a weighted $L^2$ norm of functions in $C^\infty_{\overline{X}}(S^*M)$ with a weighted $L^2$ norm of their restriction to $\partial_- S^*M$. We set $\langle \eta \rangle_h := \sqrt{1 + |\eta|_h^2}$.

**Lemma 3.1.1.** *Let $\delta < 0$. Then there exists a $C = C_\delta > 0$ such that if $u \in C^\infty_{\overline{X}}(S^*M) \cap x^\delta L^2(S^*\mathring{M}; d\lambda)$ one has, using the isomorphism* (3.1.3),

$$\frac{1}{C} \|u\|_{\langle \eta \rangle_h^{-\delta} L^2(\partial_- S^*M; d\lambda_\partial)} \le \|u\|_{x^\delta L^2(S^*\mathring{M}; d\lambda)} \le C \|u\|_{\langle \eta \rangle_h^{-\delta} L^2(\partial_- S^*M; d\lambda_\partial)} < \infty.$$

*Proof.* First note that $C^\infty_{\overline{X}}(S^*M) \cap x^\delta L^2(S^*\mathring{M}; d\lambda) \neq \emptyset$ for $\delta < 0$: indeed, let $f \in C^\infty_c(\mathring{M})$, implying that $u = If \in C^\infty_{c,\overline{X}}(S^*M) \subset C^\infty_{\overline{X}}(S^*M)$. Since $x d\lambda$ is a smooth measure on $S^*M$, one sees that $x^{-\delta} C^\infty_{\overline{X}}(S^*M) \subset L^2_{loc}(S^*M; d\lambda)$, implying the claim. Now by (3.1.4), since $u \in C^\infty_{\overline{X}}(S^*M)$, we have

$$\begin{aligned} \|u\|^2_{x^\delta L^2(S^*\mathring{M}; d\lambda)} &= \int_{S^*\mathring{M}} |x^{-\delta} u|^2 d\lambda = \int_{\partial_- S^*M} I(|u|^2 x^{-2\delta}) d\lambda_\partial \\ &= \int_{\partial_- S^*M} \left| \left( I(x^{-2\delta}) \right)^{1/2} u \right|^2 d\lambda_\partial. \end{aligned} \tag{3.1.7}$$

The second equality is valid because $|x^{-\delta} u|^2 \in L^1(S^*\mathring{M}; d\lambda)$. Let $(z, \xi) = \big((0, y), \frac{dx}{x} + \eta_\alpha dy^\alpha\big) \in \partial_- S^*M$ with $|\eta|_h > C_0 > 0$. If $C_0$ is sufficiently large, then [GGS$^+$, Lemma 2.8] implies that $x \circ \overline{\varphi}_\tau(z, \xi) = |\eta|_h^{-1} \sin(\alpha_{(z,\xi)}(\tau)) + O(|\eta|_h^{-2})$, where $\alpha_{(z,\xi)} : [0, \tau_+(z, \xi)] \to [0, \pi]$ is a family of diffeomorphisms depending smoothly on $(z, \xi) \in \partial_- S^*M$, with $\partial_\tau \alpha_{(z,\xi)}(\tau) = |\eta|_h + O(1)$, and $\tau_+(z, \xi) = |\eta|^{-1}\pi + O(|\eta|_h^{-2})$ as $|\eta|_h \to \infty$. So

$$I(x^{-2\delta})(z, \xi) = \int_0^{\tau_+(z,\xi)} x^{-1-2\delta} \circ \overline{\varphi}_\tau(z, \xi) d\tau = \int_0^{\tau_+(z,\xi)} \left( |\eta|_h^{-1} \sin(\alpha_{(z,\xi)}(\tau)) + O(|\eta|_h^{-2}) \right)^{-1-2\delta} d\tau$$

$$
\begin{aligned}
&= \int_0^\pi \left(|\eta|_h^{-1}\sin(s) + O(|\eta|_h^{-2})\right)^{-1-2\delta} \frac{ds}{|\eta|_h + O(1)} \\
&= |\eta|_h^{2\delta} \int_0^\pi \left(\sin(s) + O(|\eta|^{-1})\right)^{-1-2\delta} \frac{ds}{1 + O(|\eta|_h^{-1})}.
\end{aligned}
$$

Since $\int_0^\pi \left(\sin(s) + O(|\eta|_h^{-1})\right)^{-1-2\delta} \frac{ds}{1+O(|\eta|_h^{-1})} = a_\delta + O(|\eta|_h^{-1})$ with $a_\delta > 0$ for $\delta < 0$, we find that $I(x^{-2\delta})((z,\xi)) = a_\delta |\eta|_h^{2\delta} + O(|\eta|_h^{-1+2\delta})$ as $|\eta|_h \to \infty$. On the other hand, if $|\eta|_h \le C_0$, $I(x^{-2\delta})$ is uniformly bounded above and below by positive constants depending on $\delta$ and $C_0$. Thus (3.1.7) is comparable to $\|\langle\eta\rangle_h^\delta u\|_{L^2(\partial_- S^*M; d\lambda_\partial)} = \|u\|_{\langle\eta\rangle_h^{-\delta} L^2(\partial_- S^*M; d\lambda_\partial)}$. $\square$

## 3.2 The 0-Geometry and 0-Pseudodifferential Calculus

In this section we provide some of the background we will need later in the microlocal analysis of the operator $\mathcal{N}_g$. As already mentioned, we will use the framework of the 0-pseudodifferential calculus of Mazzeo and Melrose to construct a left inverse for $\mathcal{N}_g$ up to compact error, which together with injectivity will lead to a stability estimate. 0-pseudodifferential operators acting on functions defined on a compact manifold with boundary $M$ have Schwartz kernels that are conveniently characterized and analyzed in the 0-stretched product of Mazzeo and Melrose. This is a space obtained from $M^2$ by blowing up the boundary diagonal. In this section we define the operators of interest and describe their properties that we will need later. The main sources are [MM87] and [Maz91], also see [EMM91].

*Half Densities, the 0-and b-Tangent and Cotangent Bundles*

Since half densities will be used later in this section, we discuss them here first. Recall that given a $k$-dimensional real vector space $V$ and $\alpha \in \mathbb{R}$ one can form a 1-dimensional complex vector space $\Omega^\alpha(V)$ consisting of maps $d_\alpha : \Lambda^k(V) \setminus \{0\} \to \mathbb{C}$, called $\alpha$-densities, with the property that for $\lambda \in \mathbb{R} \setminus \{0\}$ and $\omega \in \Lambda^k(V) \setminus \{0\}$ one has $d_\alpha(\lambda\omega) = |\lambda|^\alpha d_\alpha(\omega)$. Using the functor $V \mapsto \Omega^\alpha(V)$ one can construct a line bundle on a manifold $X$, possibly with boundary or with corners[1], whose fiber at $z$ is $\Omega^\alpha(T_zX)$. We denote this bundle by $\Omega^\alpha(X)$ and denote its smooth sections by $C^\infty(X;\Omega^\alpha)$. We will be mostly interested in the case $\alpha = 1/2$ (*half density bundles*), though occasionally we will also use density bundles with $\alpha = 1$; in that case we will not write a superscript and instead write $\Omega(X)$, so there will be no confusion with the space of 1-forms, which will not be used anywhere in the text anyway. For the rest of this discussion we fix $\alpha = 1/2$. In terms of coordinates $(z^j)$, $\Omega^{1/2}(X)$ is locally trivialized by $|dz^1 \wedge \cdots \wedge dz^n|^{1/2} = |dz|^{1/2}$. Now for $X$ a smooth manifold with corners and an integer $k$ let $\Omega^{1/2}_{\{k\}}(X)$ be the smooth complex line bundle over $X$ whose smooth local sections are of the form $\prod_j x_j^{-k/2}\nu$, where $x_j$ are defining functions for the boundary hypersurfaces of $X$ and $\nu \in C^\infty(X;\Omega^{1/2})$. Note that if $M$ is the compactification of an AH manifold $(\mathring{M}^{n+1}, g)$ then $\Omega^{1/2}_{\{n+1\}}(M)$ is the geometric half density bundle, globally trivialized by $dV_g^{1/2}$. From now on we denote $\Omega^{1/2}_0(X) := \Omega^{1/2}_{\{n+1\}}(X)$, where $n+1$ is the dimension of the AH manifold of interest and for $X$ any of the manifolds with corners we will be examining. The reason for this notation is that the fiber of $\Omega^{1/2}_0(M)$ at $z \in M$ is $\Omega^{1/2}({}^0T_zM)$, where ${}^0TM$ is the smooth bundle over $M$ canonically isomorphic to $TM$ over $\mathring{M}$ and trivialized locally near $\partial M$ by $x\partial_x, x\partial_{y^1} \ldots, x\partial_{y^n}$. Here $(x, y^1, \ldots, y^n)$ are coordinates with $x$ a boundary defining function (the smooth sections of ${}^0TM$ are spanned over $C^\infty(M)$ by the 0-vector fields $\mathcal{V}_0$ discussed in the Introduction). The dual bundle of ${}^0TM$ is denoted by ${}^0T^*M$ and is trivialized locally near $\partial M$ by $dx/x, dy^1/x, \ldots, dy^n/x$. Since we will also briefly make use of elements of the b-geometry, we collect them here: for $X^d$ a manifold with corners we let $\Omega^{1/2}_b(X) = \Omega^{1/2}_{\{1\}}(X)$. Note that the fiber at $z \in X$ of $\Omega^{1/2}_b(M)$ is $\Omega^{1/2}({}^bT_zX)$, where ${}^bTX$ is the bundle whose local sections are smooth vector fields tangent to all boundary faces. That is, if $(x^1, \ldots, x^k, y^1, \ldots, y^{d-k})$ are coordinates with $x^j$ defining functions for boundary hypersurfaces then ${}^bTX$ is locally trivialized by $x^1\partial_{x^1}, \ldots, x^k\partial_{x^k}, \partial_{y^1}, \ldots, \partial_{y^{d-k}}$. Its dual is ${}^bT^*X$, trivialized by $dx^1/x^1, \ldots, dx^k/x^k, dy^1, \ldots, dy^{d-k}$ (recall that this bundle was already introduced in the special case $M = X$). For $\star \in \{\emptyset, b, 0\}$ we will write $\dot{C}^\infty(X;\Omega^{1/2}_\star)$ for

[1]We follow Melrose's convention of assuming that manifolds with corners have embedded boundary hypersurfaces; a detailed treatment of analysis on such spaces can be found in [Mel].

smooth sections of $\Omega_\star^{1/2}(X)$ whose derivatives of all orders vanish on $\partial X$ and $C^{-\infty}(X;\Omega_\star^{1/2}) := (\dot{C}^\infty(X;\Omega_\star^{1/2}))'$. We will also write $\mathcal{V}_\star$ for the space of smooth sections of ${}^\star TX$.

*Conormal and Polyhomogeneous Conormal Distributions*

It will be important later to have spaces of functions whose regularity at the boundary remains steady under the action of vector fields tangent to the boundary, and more strongly, ones that admit asymptotic expansions at the boundary. Let $X$ be a manifold with corners, with boundary hypersurfaces numbered as $X_j$, $j = 1\ldots, J$, and corresponding boundary defining functions $x^j$. For $\mathbf{s} = (s_1, \ldots, s_J) \in \mathbb{C}^J$ one defines the space of *conormal distributions of order* $\mathbf{s}$ by $\mathcal{A}^{\mathbf{s}}(X) = \{u \in C^{-\infty}(X) : x^{-\mathbf{s}} L_1 \cdots L_\ell u \in L^\infty(X), \ell \geq 0 \text{ and } L_j \in \mathcal{V}_b(X)\}$ (here $x^{\mathbf{s}} = \prod_{j=1}^J (x^j)^{s_j}$). We will refer to functions in $\bigcup_{\mathbf{s}\in\mathbb{C}^J} \mathcal{A}^{\mathbf{s}}(X)$ as conormal distributions.

We will also need the stronger notion of polyhomogeneity. Let $E \subset \mathbb{C} \times \mathbb{N}_0$ be an *index set*, that is, a discrete set with the additional properties

$$\begin{aligned} |(s_j, p_j)| \to \infty \Rightarrow \operatorname{Re}(s_j) \to \infty \text{ and} \qquad (3.2.1)\\ (s_j, p_j) \in E \Rightarrow (s_j + m, p_j - \ell) \in E, \quad m \in \mathbb{N}_0 = \{0, 1, \ldots\}, \quad \ell = 0, 1, \ldots, p_j. \qquad (3.2.2)\end{aligned}$$

If $E \subset \mathbb{C}\times\mathbb{N}_0$ satisfies (3.2.1) we will often write $\overline{E}$ to denote the smallest index set containing $E$. Now let $M$ be a manifold with boundary and $u \in C^{-\infty}(M)$. A conormal distribution $u$ is said to be *polyhomogeneous with index set* $E$ if it admits an asymptotic expansion in a collar neighborhood $[0, \varepsilon)_x \times \partial M$ of the boundary of the form

$$u \sim \sum_{(s_j,p_j)\in E} \sum_{k=0}^{p_j} x^{s_j} |\log x|^k a_{jk}(y), \quad a_{jk} \in C^\infty(\partial M). \qquad (3.2.3)$$

More precisely, the meaning of the expansion is that if $u_N$ denotes the partial sum on the right hand side of (3.2.3) restricted to $(s_j, p_j) \in E$ with $\operatorname{Re}(s_j) \leq N$ then one has $|L_1 \cdots L_\ell (u - u_N)| \leq C_{N,\ell} x^N$ for $\ell \geq 0$ and $L_j \in \mathcal{V}_b(M)$. If $u$ satisfies (3.2.3) we write $u \in \mathcal{A}^E_{phg}$. By (3.2.2), the property $u \in \mathcal{A}^E_{phg}$ does not depend on the product decomposition chosen near $\partial M$. Note that if $E_1 \subset E_2$ then $\mathcal{A}^{E_1}_{phg} \subset \mathcal{A}^{E_2}_{phg}$.

Now if $X$ is a manifold with corners with boundary hypersurfaces $X_j$, $j = 1, \dots, J$, denote by $\mathcal{E} = (E_1, \dots, E_J)$ a $J$-tuple of of index sets. The space of polyhomogeneous distributions $\mathcal{A}^{\mathcal{E}}_{phg}(X)$ is defined to be those having the form (3.2.3) with $E$ replaced by $E_j$ near the interior of the boundary hypersurface $X_j$ for $j = 1, \dots, J$ and which have product type expansions at the intersections of boundary hypersurfaces. More rigorously, $\mathcal{A}^{\mathcal{E}}_{phg}(X)$ can be defined by induction on the maximum possible codimension of boundary faces but we will not provide the details here; we refer the reader to [Maz91]. We now list a few shorthand notations related to polyhomogeneous distributions, which will be useful later. If $E$ is an index set we will write $E + \ell = \{(s + \ell, p) : (s, p) \in E\}$. The notation $\mathrm{Re}(E) > C$ will be used to denote that $\mathrm{Re}(s) > C$ for all $(s, p) \in E$. Furthermore, $\mathrm{Re}(E) \geq C$ will mean by definition that either $\mathrm{Re}(E) > C$ or $\mathrm{Re}(s) \geq C$ for all $(s, p) \in E$ and $E \cap (\{\mathrm{Re}\, z = C\} \times \{1, 2, \dots\}) = \emptyset$. Therefore $\mathrm{Re}(E) \geq 0$ suffices to guarantee that $u \in \mathcal{A}^{E}_{phg}$ is bounded. If it is known that $E \subset \mathbb{R} \times \mathbb{N}_0$ we will often be writing $E \geq C$ or $E > C$. In the special case when $u \in \mathcal{A}^{\mathcal{E}}_{phg}(X)$ is smooth down to a boundary hypersurface $X_j$ and vanishing to order $k$ there we will be replacing $E_j$ in $\mathcal{E}$ by $k$: in this case $E_j \subset \mathbb{N}_0 \times \{0\}$.

We also mention distributions conormal to an interior p-submanifold of a manifold with corners. Recall that a submanifold $Y$ of a manifold with corners $X^d$ is called a p-submanifold if for each $p \in Y$ there exists a domain $U$ of a coordinate chart for $X$ near $p$ with coordinate functions $(x, y) = (x^1, \dots, x^k, y^1, \dots, y^{d-k})$ where the $x^j$'s are defining functions of boundary hypersurfaces of $X$ and $Y \cap U$ is given as the zero set of a subset of the $x^j$, $y^\ell$. It is called an interior p-submanifold if none of the $x^j$'s vanishes identically on $Y$ (in other words, $Y$ is not contained in a boundary hypersurface of $X$). Let $Y$ be an interior p-submanifold of codimension $s$ in a $d$-dimensional manifold with corners $X$, and suppose that in coordinates $(x, y) = (x, y', y'')$ as before it is given as the zero set of $y' = (y^1, \dots, y^s)$. A distribution $u$ is said to be conormal of order $m \in \mathbb{R}$ with respect to $Y$ (denoted $u \in I^m(X, Y)$) if there exists a symbol $a \in S^{m'}\big(([0, \infty)^k \times \mathbb{R}^{d-s-k}) \times \mathbb{R}^s\big)$, $m' = m + d/4 - s/2$, such that locally

$$u(x, y) = \int_{\mathbb{R}^s} e^{iy' \cdot \xi'} a(x, y'', \xi') d\xi'. \tag{3.2.4}$$

Here $a \in S^{m'}\big(([0,\infty)^k \times \mathbb{R}^{d-s-k}) \times \mathbb{R}^s\big)$ means by definition that $a$ satisfies symbol estimates

$$|D_x^{\boldsymbol{\alpha}} D_{y''}^{\boldsymbol{\beta}} D_{\xi'}^{\boldsymbol{\gamma}} a(x, y'', \xi')| \leq C_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} \langle \xi' \rangle^{m'-|\boldsymbol{\gamma}|}, \tag{3.2.5}$$

where we use the standard notations $D = -i\partial$ and $\langle \cdot \rangle = (1+|\cdot|^2)^{1/2}$, and $\boldsymbol{\alpha} \in \mathbb{N}_0^k$, $\boldsymbol{\beta} \in \mathbb{N}_0^{d-k-s}$ and $\boldsymbol{\gamma} \in \mathbb{N}_0^s$ are multi-indices. We remark here that the name conormal for distributions satisfying (3.2.4) is justified by the fact that they have stable regularity under differentiation by vector fields tangent to $Y$; however the space where they and their derivatives lie is the somewhat cumbersome Besov space ${}^\infty H_{loc}^{-m-d/4}$, which is the reason why we prefer the definition given above. We refer the reader to Section 18.2 of [Hör07] for a detailed discussion. We will also need the space $\mathcal{A}_{phg}^{\mathcal{E}} I^m(X,Y)$, where again $X^d$ is a manifold with corners and $Y$ is an interior p-submanifold of codimension $s$: we say that $u \in \mathcal{A}_{phg}^{\mathcal{E}} I^m(X,Y)$ if $u\big|_{\mathring{X}}$ is conormal of order $m$ with respect to $Y$ and $u$ has an asymptotic expansion at each boundary face $X_i$ of $X$ of the form (3.2.3) with index set $E_i$ determined by the collection $\mathcal{E}$ and coefficients $a_{jk}$ conormal of order $m + 1/4$ with respect to $Y \cap X_i$ (here the change in the order of conormality follows Hörmander's convention; note that $\dim(X_i) = d - 1$).

If $E$ is a vector bundle over $X$ the discussion above can be used to define boundary conormal, interior conormal, polyhomogeneous, and interior conormal-boundary polyhomogeneous sections $E$, written as $\mathcal{A}^{\mathbf{s}}(X;E)$, $I^m(X,Y;E)$, $\mathcal{A}_{phg}^{\mathcal{E}}(X;E)$, and $\mathcal{A}_{phg}^{\mathcal{E}} I^m(X,Y;E)$.

*The Stretched Product*

Here we outline the construction of the 0-stretched product, i.e. the blown up space on which the Schwartz kernels of 0-pseudodifferential operators live. For a detailed exposition regarding the blow-up construction we refer the reader to [Mel] and specifically for the 0-stretched product to [MM87]. Let $M^{n+1}$ be a compact manifold with boundary as before and let $x$ be a boundary defining function; then the *0-stretched product* $M_0^2 := [M^2; \partial\Delta\iota]$ is by definition the space obtained by blowing up the boundary of the diagonal $\Delta\iota = \{(z,z) : z \in M\}$ (see Figure 3.1). More precisely, let $T_p^+ M = \{v \in T_p M : dx(v) > 0\}$ and let $SN^{++}\Delta\iota$

be the inward pointing spherical normal bundle, with fiber at $(p,p) \in \partial\Delta\iota$ given by

$$SN^{++}_{(p,p)}\partial\Delta\iota = \big(\big((\overline{T^+_p M})^2/T_{(p,p)}\partial\Delta\iota\big) \setminus o\big)/\mathbb{R}^+, \tag{3.2.6}$$

where $o$ is the 0 section. Then as a set define $M^2_0 = (M^2 \setminus \partial\Delta\iota) \bigsqcup SN^{++}\partial\Delta\iota$. There is a natural smooth structure on $M^2_0$ making it into a manifold with corners of codimension up to 3, such that the blow down map $\beta_0 : M^2_0 \to M^2$, $\beta_0\big|_{(M^2\setminus\partial\Delta\iota)} = id$, $\beta_0\big|_{SN^{++}_{(p,p)}\partial\Delta\iota} = (p,p)$ becomes smooth. Moreover, smooth vector fields tangent to $\partial\Delta\iota$ lift under the blow down map to be smooth and tangent to the boundary faces of $M^2_0$ (this is a general fact about blow-ups, see [Mel]). The set $SN^{++}(\partial\Delta\iota) \subset M^2_0$ is called the *front face* and denoted by $B_{11}$. We let $\Delta\iota_0 = \overline{\beta_0^{-1}(\Delta\iota \setminus \partial\Delta\iota)}$, which is an interior p-submanifold of $M^2_0$, transversal to the front face. We denote by $B_{10}$ the left face $\overline{\beta_0^{-1}(\partial M \times M)}$ and by $B_{01}$ the right face $\overline{\beta_0^{-1}(M \times \partial M)}$. We will occasionally refer to $B_{10}$ and $B_{01}$ as side faces and we will also be using the notation $x_{..}$ to refer to a defining function for $B_{..}$, that is, $x_{10}$ will be a defining function for $B_{10}$ and so on. Moreover, we will be writing $\mathcal{A}^{\mathcal{E}}_{phg}(M^2_0)$, $\mathcal{E} = (E_{10}, E_{01}, E_{11})$ to denote polyhomogeneous distributions on $M^2_0$ with $E_{..}$ corresponding to the face $B_{..}$.

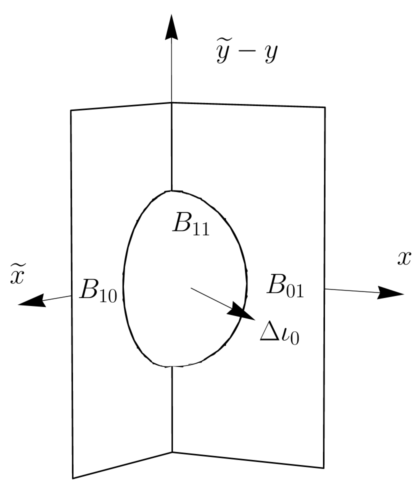


Figure 3.1: The 0-stretched product.

We now fix $p \in \partial M$. The subgroup $G_p$ of $GL(T_pM)$ that preserves $\overline{T_p^+M}$ and fixes $\partial(T_p^+M)$ pointwise induces an invariantly defined free and transitive action on $\overline{T_p^+M}$. Upon making a choice of coordinates $(x, y)$ near $p$ (with $x$ again a boundary defining function) which determine linear coordinates $(u, w)$ on $T_pM$ induced by writing $v = u\partial_x + w^\alpha \partial_{y^\alpha}$, one sees that $G_p \cong \mathbb{R}^+ \rtimes T_p\partial M \cong \mathbb{R}^+ \rtimes \mathbb{R}^n$ and the action is given by $(a, b)\cdot(u, w) = (au, w+ub)$. The group multiplication in $G_p$ is given by $(a, b) \cdot (a', b') = (aa', b' + a'b)$. The actions of $G_p^l := G_p \times Id$ and $G_p^r := Id \times G_p$ on $\left(T_p^+M\right)^2$ descend to the quotient (3.2.6) and, given linear coordinates $(u, w), (\widetilde{u}, \widetilde{w})$ on $(T_pM)^2$ chosen using two copies of the same coordinate system on $M$, we can see that each of the actions is transitive and free on the interior of each fiber $\mathring{B}_{11}\big|_p$ of the front face. Moreover, each fiber of the front face has a canonically defined singled out point $e_p$, given by $\partial\Delta\iota_0\big|_{(p,p)}$. Therefore, one obtains diffeomorphic identifications of $\mathring{B}_{11}\big|_p$ with $G_p^l \cong G_p^r \cong G_p$, hence $\mathring{B}_{11}\big|_p$ has two group structures, both of which are canonically isomorphic to $G_p$ and can be intertwined by interchanging the order of the two factors of $\left(T_p^+M\right)^2$. The diffeomorphisms $f_p^l, f_p^r : \mathring{B}_{11}\big|_p \to G_p$ obtained this way are given in terms of linear coordinates as $f_p^l\big([(u, w), (\widetilde{u}, \widetilde{w})]\big) = \big(u/\widetilde{u}, (w - \widetilde{w})/\widetilde{u}\big)$, $f_p^r\big([(u, w), (\widetilde{u}, \widetilde{w})]\big) = \big(\widetilde{u}/u, (\widetilde{w} - w)/u\big)$. Those diffeomorphisms have equivariance properties: for $q \in G_p$ write $q_l = (q, id) \in G_p^l$ and $q_r = (id, q) \in G_p^r$ to obtain as in [MM87, Section 3] that for $\omega \in \mathring{B}_{11}\big|_p$ one has

$$
\begin{aligned}
f_p^l(q_l \cdot \omega) = q \cdot f_p^l(\omega), \quad f_p^r(q_r \cdot \omega) = q \cdot f_p^r(\omega) \\
f_p^l(q_r \cdot \omega) = f_p^l(\omega) \cdot q^{-1}, \quad f_p^r(q_l \cdot \omega) = f_p^r(\omega) \cdot q^{-1}.
\end{aligned} \tag{3.2.7}
$$

By a computation in coordinates one checks that each choice of inner product on the tangent space at the identity of the Lie group $G_p$ induces a right invariant hyperbolic metric on $G_p$, so (3.2.7) implies that each such choice induces a left $G_p^r$ (resp. $G_p^l$)-invariant hyperbolic metric on $\mathring{B}_{11}\big|_p$ via $f_p^l$ (resp. $f_p^r$).

Given an AH metric $g$ and $p \in \partial M$ one obtains a canonical hyperbolic metric of curvature $-1$ on $T_p^+M$ that pulls back to $\mathring{B}_{11}\big|_p$ in two ways. If $x$ is a boundary defining function for

$\partial M$ let $h_p$ be the metric given at $v \in T_p^+ M$ by

$$h_p\big|_v := (dx(v))^{-2}\overline{g}\big|_p, \tag{3.2.8}$$

where $\overline{g} = x^2 g$ and the inner product $\overline{g}\big|_p$ on $T_pM$ is naturally identified with an inner product on $T_v(T_pM)$ for any $v \in T_p^+M$. It is easy to check that (3.2.8) does not depend on the choice of the boundary defining function $x$. The metric $h_p$ can be appropriately pulled back to $\mathring{B}_{11}\big|_p$ in two ways (that result in isometric metrics), as we explain below. Since the action of $G_p$ on $T_p^+M$ is free and transitive, given $v \in T_p^+M$ one can define a diffeomorphism $f_p^v : G_p \to T_p^+M$ by $G_p \ni q \mapsto q \cdot v \in T_p^+M$. Thus for each $v$ one obtains a hyperbolic metric $(f_p^v)^* h_p$ on $G_p$ that is right invariant with respect to the group structure of $G_p$, as one can check in coordinates. The right invariance implies that the metric $(f_p^v)^* h_p$ is in fact independent of $v$: if for $v, v' \in T_p^+M$ and $\widetilde{q} \in G_p$ one has $v = \widetilde{q} \cdot v'$, then for $q \in G_p$ $f_p^{v'}(q) = (q\widetilde{q}) \cdot v = \mathcal{R}_{\widetilde{q}}(q) \cdot v = f_p^v \circ \mathcal{R}_{\widetilde{q}}(q)$, with $\mathcal{R}_{\widetilde{q}}$ denoting right multiplication by $\widetilde{q}$ on $G_p$. Thus $(f_p^{v'})^* h_p = \mathcal{R}_{\widetilde{q}}^* (f_p^v)^* h_p = (f_p^v)^* h_p$. Hence by (3.2.7), $h_p^l := (f_p^l)^*(f_p^v)^* h_p$ and $h_p^r := (f_p^r)^*(f_p^v)^* h_p$ are hyperbolic metrics on $\mathring{B}_{11}\big|_p$ which are independent of $v \in T_p^+M$, isometric to each other by construction, and left invariant with respect to the corresponding group structure. We remark here that $d\big((f_p^l)^{-1} \circ f_p^r\big)\big|_{e_p} = -id$ as one can check in coordinates, so on $T_{e_p}(\mathring{B}_{11}\big|_p)$ one has $h_p^l = h_p^r$.

One can make a choice of coordinates to express the metric $h_p$ in (3.2.8) in a convenient form: fix a conformal representative $h_0$ in the conformal infinity of $g$ and corresponding geodesic boundary defining function $x$ and complete the gradient $\nabla^{\overline{g}} x\big|_p$ into a $\overline{g}$-orthonormal frame (where as usual $\overline{g} = x^2 g$). Then in terms of the linear coordinates $(u, w^1, \ldots, w^n)$ induced on $T_p^+M$ by this frame, (3.2.8) takes the form $h_p = u^{-2}(du^2 + |dw|^2)$. Similarly, we can express conveniently the induced metrics on $\mathring{B}_{11}\big|_p$. First, in terms of the linear coordinates above use $v = \partial_x = (1, 0) \in T_p^+M$ to construct $(f_p^v)^* h_p$ on $G_p$. Now let $\mathcal{U}' \subset T_pM$ and $\mathcal{U} \subset M$ be neighborhoods of 0 and $p$ respectively, and $\varphi : \mathcal{U}' \to \mathcal{U}$ a diffeomorphism satisfying $\varphi(0) = p$, $d\varphi\big|_0 = Id$ and $\varphi(T_p\partial M) \subset \partial M$ and consider coordinates $(x, y) = (u, w) \circ \varphi^{-1}$ near $p$. If $(\widetilde{x}, \widetilde{y})$ is a copy of $(x, y)$ on the right factor of $M^2$, $(x, y, t = \widetilde{x}/x, Y =$

$(\widetilde{y}-y)/x)$ are smooth coordinates near $B_{01}\cap B_{11}$ and away from $B_{10}$ with $(t,Y)$ coordinates on the front face, where $\varphi$ is used to identify points on $M^2$ and on $(T_p^+M)^2$ (recall the formal definition (3.2.6) of $B_{11}\big|_p$). Tracing though identifications, one sees that $h_p^r = t^{-2}(dt+|dY|^2)$ in coordinates $(t,Y)$. Analogously, using coordinates $(\widetilde{x},\widetilde{y},s=x/\widetilde{x},W=(y-\widetilde{y})/\widetilde{x})$ near $B_{10}\cap B_{11}$ and away from $B_{01}$ with $(s,W)$ coordinates on $B_{11}\big|_p$, $h_p^l = s^{-2}(ds^{\,2}+|dW|^2)$.

*The 0-Calculus*

In this section $M^{n+1}$ is a compact manifold with boundary and $x$ a boundary defining function, as usual. Throughout this chapter we will be using the 0-calculus of pseudodifferential operators of Mazzeo-Melrose ([MM87]), a class of operators generalizing and containing the approximate inverses of elliptic 0-differential operators. As already mentioned, 0-differential operators of order $m\in\mathbb{N}_0$, denoted by $\mathrm{Diff}_0^m(M)$, are the differential operators that can be written as finite sums of at most $m$-fold products of vector fields in $\mathcal{V}_0$: that is, near $\partial M$ one has in coordinates as before

$$P=\sum_{j+|\boldsymbol{\alpha}|\leq m} a_{j,\boldsymbol{\alpha}}(x,y)(x\partial_x)^j(x\partial_y)^{\boldsymbol{\alpha}},\quad a_{j,\boldsymbol{\alpha}}\in C^\infty(M)$$

where we use multi-index notation.

By the Schwartz kernel theorem, operators $P:\dot{C}^\infty(M;\Omega_0^{\frac{1}{2}})\to C^{-\infty}(M;\Omega_0^{\frac{1}{2}})$ are in one to one correspondence with kernels $\kappa_P\in C^{-\infty}(M^2;\Omega_0^{1/2})$ (note that $\pi_l^*\Omega_0^{1/2}(M)\otimes\pi_r^*\Omega_0^{1/2}(M)\cong\Omega_0^{1/2}(M^2)$, where $\pi_l,\pi_r:M^2\to M$ denote left and right projection respectively). The Schwartz kernels of operators in the 0-calculus are naturally described in $M_0^2$. As shown in [MM87], smooth sections of $\Omega_0^{1/2}(M^2)$ lift via $\beta_0$ to smooth sections of $\Omega_0^{1/2}(M_0^2)$ and hence for any kernel we have $\beta_0^*\kappa_P\in C^{-\infty}(M_0^2;\Omega_0^{1/2})$. We begin by defining the small 0-calculus of order $m$: let

$$\Psi_0^m(M)\ni P:\dot{C}^\infty(M;\Omega_0^{1/2})\to C^{-\infty}(M;\Omega_0^{1/2})$$

be any operator whose Schwartz kernel $\kappa_P$ satisfies $\beta_0^*\kappa_P\in\mathcal{A}_{phg}^{\mathcal{E}}I^m(M_0^2,\Delta\iota_0;\Omega_0^{1/2})$ with $\mathcal{E}=\big(\emptyset,\emptyset,\overline{\{(0,0)\}}\big)$, so $\kappa_P$ is a section of $\Omega_0^{1/2}(M_0^2)$ conormal of order $m$ to $\Delta\iota_0$, smooth down to the front face away from $\Delta\iota_0$ and vanishing to infinite order at the side faces. We set $\Psi_0^{-\infty}(M) = \bigcap_{m\in\mathbb{R}} \Psi_0^m(M)$. Note that the kernel of an element in $\Psi_0^m(M)$ is given by an oscillatory integral. This implies that unless the corresponding symbol (see (3.2.4)) is integrable the kernel is not defined pointwise, however it makes sense as a distribution via consecutive formal integrations by parts; see, for instance Theorem 1.11 in [GS94]. In our case, by following the proof of this theorem we see that this amounts to the fact that given an operator $P \in \Psi_0^m(M)$ and $N \in \mathbb{N}_0$ we can write

$$P = \sum_{j=0}^{M} P_j Q_j + \widetilde{P}, \quad P_j \in \Psi_0^{m-N}(M), \quad Q_j \in \mathrm{Diff}_0^N(M), \quad \widetilde{P} \in \Psi_0^{-\infty}(M). \tag{3.2.9}$$

As hinted by (3.2.4), to any operator $P \in \Psi_0^m(M)$ corresponds a principal symbol encoding the leading conormal singularity on $\Delta\iota_0$: one has

$$\sigma_0^m(\kappa_P) \in S^m(N^*\Delta\iota_0; \Omega_0^{1/2}(M_0^2)\big|_{N^*\Delta\iota_0} \otimes \Omega_{fiber}(N^*\Delta\iota_0))/S^{m-1};$$

that is, the symbol is a symbolic section[2] of the bundle $\Omega_0^{1/2}(M_0^2)\big|_{N^*\Delta\iota} \otimes \Omega_{fiber}(N^*\Delta\iota) \cong \Omega_0(M) \otimes \Omega_{fiber}(N^*\Delta\iota)$. Here $\Omega_{fiber}(N^*\Delta\iota)$ is the density on the fibers of $N^*(\Delta\iota)$ and it arises from computing the invariant Fourier transform (see [Sim90]) of the kernel on the fibers of $N^*\Delta\iota_0$. Using the canonical identification of $N^*\Delta\iota_0 \leftrightarrow {}^0T^*M$ it can be shown (see [MM87], [Lau03]) that $\Omega_0(M)\otimes\Omega_{fiber}(N^*\Delta\iota) \cong \Omega({}^0T^*M)$, which is canonically trivial, hence $\sigma_0^m\kappa_P)$ can be identified with a symbol $\sigma_0^m(P) \in S^{\{m\}}({}^0T^*M) := S^m({}^0T^*M)/S^{m-1}$, which we call the *principal symbol*. Provided there exists a symbol $a \in S^{\{-m\}}({}^0T^*M)$ such that $\sigma_0^m(P)\cdot a \equiv 1$, $P$ will be called *elliptic*.

We further define the space of operators whose kernels are smooth in $(M_0^2)^\circ$ with polyhomogeneous expansions at the boundary faces: $P \in \Psi_0^{-\infty,\mathcal{E}}(M) \iff \beta_0^*\kappa_P \in \mathcal{A}^{\mathcal{E}}_{phg}(M_0^2;\Omega_0^{1/2})$, $\mathcal{E} = (E_{10}, E_{01}, E_{11})$. We finally define the large 0-calculus as the operators with kernels satisfying $\beta_0^*\kappa_P \in \mathcal{A}^{\mathcal{E}}_{phg} I^m(M_0^2, \Delta\iota_0; \Omega_0^{1/2})$, for $\mathcal{E}$ as before and $m \in \mathbb{R}$. We will often write $\Psi_0^{m,E_{10},E_{01}}(M)$ to imply that $E_{11} = \overline{\{(0,0)\}}$. Note that in this case using a cutoff function

[2] That is, the symbolic estimates (3.2.5) hold with $N^*\Delta\iota_0$ identified locally near $B_{11}\cap\Delta\iota_0$ with $[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^{n+1}$.

supported near $\Delta\iota_0$ one sees that

$$\Psi_0^{m,E_{10},E_{01}}(M) = \Psi_0^m(M) + \Psi_0^{-\infty,E_{10},E_{01}}(M). \tag{3.2.10}$$

The rest of the shorthand notations for index sets outlined earlier will apply for $\Psi_0^{m,\mathcal{E}}$; for instance $P \in \Psi_0^{m,a,E_{01},E_{11}}(M)$, $a \in \mathbb{N}_0$, indicates that $\beta_0^* K_P$ is smooth near the interior of $B_{10}$ and vanishes at $B_{10}$ to order $a$.

The definition of 0-pseudodifferential operators does not depend on the existence of a metric; however since this is the setting in which we will use them, we fix an AH metric $g$ on $\mathring{M}$ so that $\Omega_0^{1/2}(M)$ is canonically trivialized by $\gamma_0 = dV_g^{1/2}$, and we write $\kappa_P = K_P(z,\widetilde{z}\,) \cdot \gamma_0(z) \otimes \gamma_0(\widetilde{z})$ for $P \in \Psi_0^{m,\mathcal{E}}(M)$. To clarify the action of $P$ in terms of coordinates near $B_{11}$, we choose copies $(x,y)$, $(\widetilde{x},\widetilde{y})$ of the same coordinate system on the two factors of $M^2$ near a point $(p,p) \in \partial\Delta\iota$ as before. As already mentioned, $(x,y,t=\widetilde{x}/x, Y=(\widetilde{y}-y)/x)$ are smooth coordinates in a neighborhood of $B_{11}\setminus B_{10}$ and away from $B_{10}$; in terms of them, $t$ is a defining function for $B_{01}$ and $x$ is a defining function for $B_{11}$. On the other hand, $(\widetilde{x},\widetilde{y}, s = x/\widetilde{x}, W = (y-\widetilde{y})/\widetilde{x})$ are valid coordinates away from $B_{01}$ and in terms of them $s$ is a defining function for $B_{10}$. We use the notations $\beta_0^* K_P^r$, $\beta_0^* K_P^l$ for $\beta_0^* K_P$ expressed in terms of coordinates $(x,y,t,Y)$ and $(\widetilde{x},\widetilde{y},s,W)$ respectively. Then for $P \in \Psi_0^{m,\mathcal{E}}(M)$ and $f \in \dot{C}^\infty(M)$ we have

$$P(f\cdot\gamma_0)(x,y) = \int \beta_0^* K_P^l\Big(\frac{x}{s}, y - \frac{W}{s}x, s, W\Big) f\Big(\frac{x}{s}, y - \frac{W}{s}x\Big)\Big(\det \overline{g}\Big(\frac{x}{s}, y - \frac{W}{s}x\Big)\Big)^{\frac{1}{2}} \frac{|dsdW|}{s}\cdot\gamma_0. \tag{3.2.11}$$

We interpret the action of a differential operator $P \in \mathrm{Diff}_0^m(M)$ on $f\cdot\gamma_0$ as $P(f\cdot\gamma_0) = (Pf)\cdot\gamma_0$.

Operators in the large 0-calculus can be composed under compatibility assumptions. The following proposition follows from Theorem 3.15 in [Maz91] with a change in normalizations. In this form it can also be found in [Alb].

**Proposition 3.2.1.** *Let $P \in \Psi_0^{m,\mathcal{E}}(M)$, $P' \in \Psi_0^{m',\mathcal{F}}(M)$. If $\mathrm{Re}(E_{01}+F_{10}) > n$ then the composition $P \circ P'$ is defined and $P \circ P' \in \Psi_0^{m+m',\mathcal{W}}(M)$, where $\mathcal{W}$ is given by*

$$W_{10} = (F_{10}+E_{11})\overline{\cup}E_{10}, \quad W_{01} = (E_{01}+F_{11})\overline{\cup}F_{01},\; W_{11} = (E_{10}+F_{01})\overline{\cup}(E_{11}+F_{11});$$

*here the sum and extended union respectively of the index sets $E$, $E'$ are given by*

$$\begin{aligned} E + E' =& \{(s,p) + (s',p') : (s,p) \in E,\ (s',p') \in E'\}, \\ E \overline{\cup} E' =& E \cup E' \cup \{(s, p+p'+1) : \textit{there exist } (s,p) \in E, (s,p') \in E'\}. \end{aligned}$$

We also state results regarding mapping properties on polyhomogeneous functions and on half densities in Sobolev spaces. The following can be proved using Melrose's Push-forward Theorem (see Theorem 3.2.3 below), see [Maz91] and [Alb]:

**Proposition 3.2.2.** *Let $u \in \mathcal{A}^F_{phg}(M; \Omega_0^{1/2})$ and $P \in \Psi_0^{m,\mathcal{E}}(M)$, $m \in \mathbb{R}$. If $\operatorname{Re}(E_{01} + F) > n$ then $Pu \in \mathcal{A}^{F'}_{phg}(M; \Omega_0^{1/2})$, where $F' = E_{10} \overline{\cup} (E_{11} + F)$.*

The next proposition contains mapping properties for elements in the large 0-calculus in terms of weighted Sobolev half densities, denoted by $x^\delta H_0^s(M; \Omega_0^{1/2})$. We remark here that $C_c^\infty(M; \Omega_0^{1/2})$ (and thus also $\dot{C}^\infty(M; \Omega_0^{1/2})$) is dense in $x^\delta H_0^s(M; \Omega_0^{1/2})$ for $s \geq 0$ (see Lemma 3.9 in [Lee06]) and the inclusion $x^{\delta'} H_0^{m'}(M; \Omega_0^{1/2}) \hookrightarrow x^\delta H_0^m(M; \Omega_0^{1/2})$ is compact provided $m' > m$ and $\delta' > \delta$.[3] Proposition 3.2.4 below follows from Corollary 3.23 and Theorem 3.25 in [Maz91] upon taking into account the difference in conventions regarding the definition of 0-pseudodifferential operators and the different densities on which they act; for completeness we outline part of the proof following the exposition in [Alb], where Melrose's Push-forward Theorem is used. An earlier version of this result with our convention also appears in [Maz86], Corollary 2.53, and another appears in [Alb], though the assumptions regarding the weights there are stronger than they need to be. We first state the Push-forward Theorem.

Let $X$, $Y$ be manifolds with corners with embedded boundary hypersurfaces $X_j$, $Y_i$ respectively, $j = 1, \ldots, J$, $i = 1, \ldots, J'$, and let $\rho_j$, $r_i$ be corresponding defining functions. Then a smooth map $f : X \to Y$ is an *interior b-map* if for each $i$ $f^* r_i = h \prod_j \rho_j^{e(i,j)}$, $e(i,j) \in \mathbb{N}_0$ where $h \in C^\infty(X)$ is non-vanishing. For such $f$ the differential extends by continuity from the interior to define the *b-differential* ${}^b f_* : {}^b T_z X \to {}^b T_{f(z)} Y$, $z \in X$. The

[3]Note that the subscript 0 in $H_0^k$ has to do with the 0-vector fields that generate its norm, and the spaces $H_0^k(M; dV_g)$ should not be confused with the Sobolev spaces $H_0^k(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open (i.e. the closure of $C_c^\infty(\Omega)$ in the $H^k(\Omega)$ norm), that will not be used anywhere in this chapter.

interior b-map $f$ is called a *b-fibration* if its b-differential is everywhere surjective and in addition for each $j$ there exists at most one $i$ such that $e(i,j) \neq 0$ (this means that $f$ maps no boundary hypersurface into a corner).

**Theorem 3.2.3** ([Mel]). *Let $X$, $Y$ be manifolds with corners with embedded boundary hypersurfaces $X_j$, $Y_i$, and $f : X \to Y$ a b-fibration as before. Also let $\mathcal{E} = (E_1, \dots E_J)$ be an index family for $X$ such that $E_j$ corresponds to $X_j$. If $u \in \mathcal{A}^{\mathcal{E}}_{phg}(X; \Omega_b)$ and $E_j > 0$ for each $j$ such that $e(i,j) = 0$ for all $i$ (that is, for each $j$ such that $X_j$ is not mapped into $\partial Y$), then*

$$f_* u \in \mathcal{A}^{f_\#(\mathcal{E})}_{phg}(Y; \Omega_b),$$

*where $f_\#(\mathcal{E}) = (F_1, \dots, F_{J'})$ is the index family defined by*

$$F_i = \overline{\bigcup}_{j : e(i,j) \neq 0} \left( \left( \frac{s}{e(i,j)}, p \right) : (s,p) \in E_j \right),$$

*where $F_i$ corresponds to $Y_i$.*

Then one has

**Proposition 3.2.4.** *Let $P \in \Psi^{m,\mathcal{E}}_0(M)$, $m \in \mathbb{R}$. Provided $s \in \mathbb{R}$, $\mathrm{Re}(E_{01}) > n/2 - \delta$, $\mathrm{Re}(E_{11}) \geq \delta' - \delta$ and $\mathrm{Re}(E_{10}) > \delta' + n/2$ one has that*

$$P : x^\delta H^s_0(M; \Omega^{1/2}_0) \to x^{\delta'} H^{s-m}_0(M; \Omega^{1/2}_0)$$

*is bounded. In particular, if $m < 0$, $\mathrm{Re}(E_{01}) > n/2 - \delta$, $\mathrm{Re}(E_{11}) > 0$ and $\mathrm{Re}(E_{10}) > \delta + n/2$ then $P : x^\delta H^s_0(M; \Omega^{1/2}_0) \to x^\delta H^s_0(M; \Omega^{1/2}_0)$ is compact.*

*Proof.* We will show that $P : x^\delta L^2(M; \Omega^{1/2}_0) \to x^{\delta'} L^2(M; \Omega^{1/2}_0)$ for $P \in \Psi^{-\infty,\mathcal{E}}_0(M)$ and $\delta$, $\delta'$ as in the statement, since this is the only step of the proof that differs from the presentation in [Alb], and refer the reader there for the general statement. Let $\beta_r = \pi_l \circ \beta_0$, $\beta_l = \pi_l \circ \beta_0$, where $\pi_l$, $\pi_r$ are projections onto the right and left factors of $M^2$ respectively. The maps $\beta_l$, $\beta_r$ are b-fibrations and we have $\beta_l^* x = x_{10} x_{11}$ and $\beta_r^* \widetilde{x} = x_{01} x_{11}$ (as usual $x$, $\widetilde{x}$ are boundary defining functions for $\partial M \times M$, $M \times \partial M$ respectively). Recalling that $\gamma_0 = dV_g^{1/2}$, by [MM87, Lemma 4.6] we have that $\nu_0 := \beta_l^*\gamma_0 \otimes \beta_r^*\gamma_0 \in C^\infty(M_0^2;\Omega_0^{1/2})$; set $\nu_b := (x_{10}x_{11}x_{01})^{n/2}\nu_0 \in C^\infty(M_0^2;\Omega_b^{1/2})$. As before, we write $\kappa_P = K_P(z,\widetilde{z}\,)\cdot\gamma_0(z)\otimes\gamma_0(\widetilde{z})$ for the Schwartz kernel of $P$; recall that $K_P$ is smooth away from $\partial\Delta\iota$. For $u' = u\cdot\gamma_0 \in x^\delta L^2(M;\Omega_0^{1/2})$, $v' = v\cdot\gamma_0 \in x^{-\delta'}L^2(M;\Omega_0^{1/2}) = (x^{\delta'}L^2(M;\Omega_0^{1/2}))'$ we compute, using Cauchy-Schwarz,

$$\begin{aligned}|(v',Pu')| &\leq \int_{M^2} |x^{\delta'}v(z)|\left(\frac{\widetilde{x}}{x}\right)^{n/4}|K_P(z,\widetilde{z})|\left(\frac{x}{\widetilde{x}}\right)^{n/4}|\widetilde{x}^{-\delta}u(\widetilde{z})|\widetilde{x}^{\delta}x^{-\delta'}\gamma_0^2(z)\otimes\gamma_0^2(\widetilde{z}) \quad (3.2.12)\\ &\leq \left(\int_{M^2}|x^{\delta'}v(z)|^2\left(\frac{\widetilde{x}}{x}\right)^{n/2}|K_P(z,\widetilde{z})|\widetilde{x}^{\delta}x^{-\delta'}\gamma_0^2(z)\otimes\gamma_0^2(\widetilde{z})\right)^{1/2}\\ &\qquad\times\left(\int_{M^2}|K_P(z,\widetilde{z})|\left(\frac{x}{\widetilde{x}}\right)^{n/2}|\widetilde{x}^{-\delta}u(\widetilde{z})|^2\widetilde{x}^{\delta}x^{-\delta'}\gamma_0^2(z)\otimes\gamma_0^2(\widetilde{z})\right)^{1/2}\end{aligned}$$

It now suffices to show

$$(\pi_l)_*\left(\left(\frac{\widetilde{x}}{x}\right)^{n/2}|K_P|\widetilde{x}^{\delta}x^{-\delta'}\gamma_0^2(z)\otimes\gamma_0^2(\widetilde{z})\right)\in L^\infty(M;\Omega_0), \quad (3.2.13)$$

$$(\pi_r)_*\left(\left(\frac{x}{\widetilde{x}}\right)^{n/2}|K_P|\widetilde{x}^{\delta}x^{-\delta'}\gamma_0^2(z)\otimes\gamma_0^2(\widetilde{z})\right)\in L^\infty(M;\Omega_0). \quad (3.2.14)$$

By Theorem 3.2.3, one has

$$\begin{aligned}&(\pi_l)_*\left(\left(\frac{\widetilde{x}}{x}\right)^{n/2}|K_P|\widetilde{x}^{\delta}x^{-\delta'}\gamma_0^2(z)\otimes\gamma_0^2(\widetilde{z})\right)\\ &\qquad= (\beta_l)_*\left(\left(\frac{x_{01}}{x_{10}}\right)^{n/2}|\beta_0^*K_P|x_{10}^{-\delta'}x_{11}^{\delta-\delta'}x_{01}^{\delta}(x_{10}x_{11}x_{01})^{-n}\nu_b^2\right)\in\mathcal{A}^F_{phg}(M;\Omega_b),\end{aligned}$$

where $F = (E_{11}+\delta-\delta'-n)\overline{\cup}(E_{10}-n-n/2-\delta')$, provided $\mathrm{Re}(E_{01}) > -\delta+n/2$. Since $\mathcal{A}^F_{phg}(M;\Omega_b)$ can be identified with $\mathcal{A}^{F'}_{phg}(M;\Omega_0)$, where $F' = F+n$, (3.2.13) is true provided $\mathrm{Re}(E_{11}) \geq -\delta+\delta'$ and $\mathrm{Re}(E_{10}) \geq n/2+\delta'$. Similarly, provided $\mathrm{Re}(E_{10}) > \delta'+n-n/2$

$$\begin{aligned}&(\pi_r)_*\left(\left(\frac{x}{\widetilde{x}}\right)^{n/2}|K_P|\widetilde{x}^{\delta}x^{-\delta'}\gamma_0(z)\otimes\gamma_0(\widetilde{z})\right)\\ &\qquad= (\beta_r)_*\left(\left(\frac{x_{10}}{x_{01}}\right)^{n/2}|\beta_0^*K_P|x_{10}^{-\delta'}x_{11}^{\delta-\delta'}x_{01}^{\delta}(x_{10}x_{11}x_{01})^{-n}\nu_b^2\right)\in\mathcal{A}^{\widetilde{F}}_{phg}(M;\Omega_b),\end{aligned}$$

with $\widetilde{F} = (E_{11}+\delta-\delta'-n)\overline{\cup}(E_{01}-n/2+\delta-n)$. As before, (3.2.14) holds if $\mathrm{Re}(E_{11}) \geq -\delta+\delta'$ and $\mathrm{Re}(E_{01}) \geq n/2-\delta$. Note that the device of multiplying and dividing by $\left(\frac{x}{\widetilde{x}}\right)^{n/4}$ in (3.2.12), similarly to the proof of Theorem 3.25 in [Maz91], allows us to extend the range of $\delta$, $\delta'$ for which the result holds. $\square$

*The Model Operator*

Let $p \in \partial M$, where $M^{n+1}$ is a compact manifold with boundary. Any operator $P \in \Psi_0^{m,\mathcal{E}}(M)$ with $\mathrm{Re}(E_{11}) \geq 0$ determines an invariantly defined operator $N_p(P) : C_c^\infty(T_p^+ M; \Omega_0^{1/2}) \to C^{-\infty}(T_p^+ M; \Omega_0^{1/2})$, which will be called the *model operator*[4], and which captures the leading order behavior of the Schwartz kernel of $P$ at the front face. Here $C^{-\infty}(T_p^+ M; \Omega_0^{1/2}) = \left(C_c^\infty(T_p^+ M; \Omega_0^{1/2})\right)'$. The model operator is closely related to the group structures on the front face discussed earlier and can be defined independently of the existence of a metric on $M$. If $\mathring{M}$ is endowed with an AH metric $g$ and $P$ is an operator depending on $g$ it often happens that $N_p(P)$ can be identified with the corresponding operator on hyperbolic space: for instance the model operator of the Laplacian of an AH metric is the hyperbolic Laplacian on $(T_p^+ M, h_p)$ (see (3.2.8)), as shown in [Maz91]. We will show an analogous result for the normal operator $\mathcal{N}_g$ in Proposition 3.4.1. Below we assume for simplicity that an AH metric $g$ is fixed, so that the various density bundles are canonically trivial.

Let $\mathcal{U}' \subset T_p M$ and $\mathcal{U} \subset M$ be neighborhoods of 0 and $p$ respectively, and $\varphi : \mathcal{U}' \to \mathcal{U}$ a diffeomorphism with the properties $\varphi(0) = p$, $d\varphi\big|_0 = Id$ and $\varphi(T_p \partial M) \subset \partial M$. Also let $R_r : T_p^+ M \to T_p^+ M$, $r \in (0, \infty)$, be the canonical radial action and $\gamma_0^{h_p} = dV_{h_p}^{1/2}$, where $h_p$ is given by (3.2.8). Then if $P \in \Psi_0^{m,\mathcal{E}}(M)$ with $\mathrm{Re}(E_{11}) \geq 0$ and $f \cdot \gamma_0^{h_p} \in C_c^\infty(T_p^+ M; \Omega_0^{1/2})$ the model operator is defined by

$$N_p(P)(f \cdot \gamma_0^{h_p}) = \lim_{r \to 0} R_r^* \varphi^* P (\varphi^{-1})^* R_{1/r}^* (f \cdot \gamma_0^{h_p}). \tag{3.2.15}$$

It can be shown (see, for instance [MM87]) that $N_p(P)$ is independent of the choice of $\varphi$ satisfying the properties listed above. If $P = \sum_{j,\alpha} a_{j,\alpha}(x,y)(x\partial_x)^j (x\partial_y)^\alpha \in \mathrm{Diff}_0^k(M)$ its model operator has a simple expression in terms of linear coordinates $(u, w)$ on $T_p^+ M$ induced

[4]As already mentioned, the more common name for the model operator is normal operator. Despite not following the usual convention for its name, we maintain the traditional notation $N_p$.

by coordinates $(x,y)$ on $M$: one has $N_p(P) = \sum_{j,\alpha} a_{j,\alpha}(0,0)(u\partial_u)^j(u\partial_w)^\alpha$, that is, $N_p(P)$ is given by "freezing coefficients" at $p$.

As we already mentioned, each fiber $\mathring{B}_{11}\big|_p$ of the front face carries two group structures isomorphic to the group $G_p \subset GL(T_pM)$ and hence the front face acts on $T_p^+M$ from the left. So given a distribution on $u \in C^{-\infty}(\mathring{B}_{11}\big|_p)$ one can define an operator on $T_p^+M$ by left convolution; i.e. by

$$u * f(v) := \int u(q) f(q^{-1}\cdot v)\, dH(q), \quad f \in C_c^\infty(T_p^+M),$$

where $dH$ is a left invariant Haar measure on $\mathring{B}_{11}\big|_p$ (which is determined up to scaling). A choice of an AH metric on $\mathring{M}$ determines a preferred Haar measure, since it determines an inner product on $T_{e_p}(\mathring{B}_{11}\big|_p)$ as discussed earlier. Recall that the kernel of an operator in $\Psi_0^{m,\mathcal{E}}(M)$, $\mathrm{Re}(E_{11}) \geq 0$, is of the form $\kappa_P = K_P(z,\widetilde{z}) \cdot \gamma_0(z) \otimes \gamma_0(\widetilde{z})$, where $\beta_0^* K_P$ is continuous down to the front face with values in distributions conormal to the lifted diagonal $\Delta\iota_0$. Since $\Delta\iota_0$ is transversal to $B_{11}\big|_p$, $P$ determines in a natural way a distribution on $B_{11}\big|_p$ by restriction: one has

$$F_p(P) := \beta_0^* K_P\big|_{B_{11}|_p} \in \mathcal{A}_{phg}^{E_{10},E_{01}} I^{m+\frac{n+1}{4}}(B_{11}\big|_p, \{e_p\}), \tag{3.2.16}$$

where $E_{10}$, $E_{01}$ correspond to expansions at $B_{10}\cap B_{11}$ and $B_{01}\cap B_{11}$ respectively and the order of conormality follows Hörmander's convention, described earlier. By [MM87, Proposition 5.19], for operators with smooth kernel down to the interior of the front face there exists a short exact sequence

$$0 \hookrightarrow \Psi_0^{-\infty,E_{10},E_{01},1}(M) \to \Psi_0^{-\infty,E_{10},E_{01}}(M) \xrightarrow{F_*} \mathcal{A}_{phg}^{E_{10},E_{01}}\left(B_{11}\big|_*\right) \to 0, \tag{3.2.17}$$

where $\mathcal{A}_{phg}^{E_{10},E_{01}}\left(B_{11}\big|_*\right)$ is a family distributions depending parametrically and smoothly on a boundary point and an AH metric has been used to identify the kernels of operators with functions.

The operator on $T_p^+M$ given by convolution by the kernel (3.2.16) agrees with the model operator of $P$:

**Lemma 3.2.5.** *Let $P \in \Psi_0^{m,\mathcal{E}}(M)$ with $\mathrm{Re}(E_{11}) \geq 0$, where $(M,g)$ is AH. Then for each $p \in \partial M$ and $f \cdot \gamma_0^{h_p} \in C_c^\infty(T_p^+ M; \Omega^{1/2})$ one has $N_p(P)(f \cdot \gamma_0^{h_p}) = (F_p(P) * f) \cdot \gamma_0^{h_p}$ where the convolution is with respect to the Haar measure determined by $g$.*

*Proof.* Let $(u,w)$ be linear coordinates on $T_p M^+$ with $u$ boundary defining function and let $\varphi : \mathcal{U}' \to \mathcal{U}$ be as above, so that $(x,y) = (u,w) \circ \varphi^{-1}$ become coordinates on $M$ near $p$ with $x$ boundary defining function. By (3.2.15) it follows that for $f \in C_c^\infty(T_p^+ M)$ and $r$ small $\operatorname{supp} R_{1/r}^* f \subset \mathcal{U}'$ and thus $\operatorname{supp}\big((\varphi^{-1})^* R_{1/r}^* f\big) \subset \mathcal{U}$, where the coordinates $(x,y)$ are valid. Hence by (3.2.11) and (3.2.15) we find (upon identifying $(x,y)$ with $(u,w)$)

$$
\begin{aligned}
N_p(P)(f \cdot \gamma_0^{h_p})(u,w) =& \lim_{r \to 0} \int \beta_0^* K_P^l\Big(r\frac{u}{s}, rw - \frac{W}{s} ru, s, W\Big) \\
& \times f\Big(\frac{u}{s}, w - \frac{W}{s}u\Big) \sqrt{\det \overline{g}(0,0)} \frac{|dsdW|}{s} \cdot \frac{\sqrt{\det \overline{g}(ru,rw)}}{\sqrt{\det \overline{g}(0,0)}} \gamma_0^{h_p} \\
=& \int \beta_0^* K_P^l(0,0,s,W) f\Big(\frac{u}{s}, w - \frac{W}{s}u\Big) \sqrt{\det \overline{g}(0,0)} \frac{|dsdW|}{s} \cdot \gamma_0^{h_p} \\
=& \int \beta_0^* K_P^l\big(0,0,(s,W)\big) f\big((s,W)^{-1} \cdot (u,w)\big) \sqrt{\det \overline{g}(0,0)} \frac{|dsdW|}{s} \cdot \gamma_0^{h_p} \\
=& (F_p(P) * f)(u,w) \cdot \gamma_0^{h_p}.
\end{aligned}
\tag{3.2.18}
$$

Here $\sqrt{\det \overline{g}(0,0)} s^{-1} |dsdW|$ is the left invariant Haar measure on $\mathring{B}_{11}\big|_p$ induced by the metric $h_p$ as described earlier. $\square$

**Remark 3.2.6.** *It follows from Lemma 3.2.5 that if $\chi \in C_c^\infty(\mathcal{U})$ is identically 1 in a neighborhood of $p$ then $N_p(P\chi) = N_p(\chi P) = N_p(P)$.*

Given a choice of coordinates near $p$ such that $h_p = u^{-2}(du^2 + |dw|^2)$, $(T_p^+ M, h_p)$ can be isometrically identified with $\mathbb{H}^{n+1} = \{(u,w) \in \mathbb{R}^+ \times \mathbb{R}^n\}$, the hyperbolic upper half space, and the same is true for $\mathring{B}_{11}\big|_p$ as already discussed. Using those identifications we can regard $N_p(P)$ as an operator on $\mathbb{H}^{n+1}$ and rewrite (3.2.18) as

$$
N_p(P) f(u,w) = \int \beta_0^* K_P^l\Big(0,0,\frac{u}{\widetilde{u}}, \frac{w - \widetilde{w}}{\widetilde{u}}\Big) f(\widetilde{u}, \widetilde{w}) \frac{|d\widetilde{u} d\widetilde{w}|}{\widetilde{u}^{n+1}}. \tag{3.2.19}
$$

Note that $\beta_0^* K_P^l\big(0,0,\frac{u}{\widetilde{u}},\frac{w-\widetilde{w}}{\widetilde{u}}\big) = \beta_0^* K_P^r\big(0,0,\frac{\widetilde{u}}{u},\frac{\widetilde{w}-w}{u}\big)$ and the former is polyhomogeneous with index set $E_{10}$ in $u/\widetilde{u}$ while the latter is polyhomogeneous with index set $E_{01}$ in $\widetilde{u}/u$. This implies that if $P \in \Psi_0^{m,\mathcal{E}}(M)$ with $\mathrm{Re}(E_{11}) \geq 0$, over compact subsets of $\mathbb{H}^{n+1}$ one has $K_{N_p(P)} \in \mathcal{A}_{phg}^{\mathcal{E}'} I^m((\overline{\mathbb{H}^{n+1}})_0^2; \Delta\iota_0)$ with $\mathcal{E}' = (E_{10}, E_{01}, \overline{\{(0,0)\}})$. Conjugating by the Cayley transform, $N_p(P)$ can be interpreted as an operator on $\overline{\mathbb{B}^{n+1}}$ and one has $N_p(P) \in \Psi_0^{m,E_{10},E_{01}}(\overline{\mathbb{B}^{n+1}})$; thus the model operator also extends to appropriate weighted Sobolev spaces according to Proposition 3.2.4.

We now prove the following proposition stating that under suitable assumptions the model operator is a homomorphism. It is stated and proved in [MM87] in the case $P \in \mathrm{Diff}_0^m(M)$. It is also mentioned in [Alb] and in [EMM91] that the homomorphism property holds, though without explicit mention of hypotheses that need to be assumed.

**Proposition 3.2.7.** *Let $P$, $P'$ be as in Proposition 3.2.1, with the additional assumptions* $\mathrm{Re}(E_{11}) \geq 0$, $\mathrm{Re}(F_{11}) \geq 0$ *and* $\mathrm{Re}(E_{10} + F_{01}) > 0$. *Then for each $p \in \partial M$ one has $N_p(P \circ P') = N_p(P) \circ N_p(P')$.*

**Remark 3.2.8.** *The assumptions* $\mathrm{Re}(E_{11}) \geq 0$, $\mathrm{Re}(F_{11}) \geq 0$ *and* $\mathrm{Re}(E_{10}+F_{01}) > 0$ *guarantee that $P$, $P'$ and $P \circ P'$ have well defined model operators (see Proposition 3.2.1).*

*Proof.* Without loss of generality we can assume that $E_{11} = F_{11} = \overline{\{(0,0)\}}$: indeed, upon restricting $\beta_0^* K_P$ (resp. $\beta_0^* K_{P'}$) at $B_{11}\big|_p$, any term in the asymptotic expansion of $\beta_0^* K_P$ (resp. $\beta_0^* K_{P'}$) at $B_{11}\big|_p$ corresponding to $(s_j, p_j) \in E_{11}$ (resp. $F_{11}$) with $\mathrm{Re}(s_j) > 0$ does not contribute to the model operator of $P$ (resp. $P'$). By Proposition 3.2.1 and the assumption $\mathrm{Re}(E_{10} + F_{01}) > 0$ any such coefficient does not contribute to the model operator of the composition $P \circ P'$ either.

By [MM87, Proposition 5.19] the claim holds if $P \in \mathrm{Diff}_0^m(M)$ and $P' \in \Psi_0^{m',\mathcal{F}}(M)$, and it also follows very similarly if $P' \in \mathrm{Diff}_0^{m'}(M)$ and $P \in \Psi_0^{m,\mathcal{F}}(M)$ (or if both $P$, $P'$ are differential). Using this fact, we observe that it suffices to show the claim for $m$, $m' < N_0$, where $N_0 \in \mathbb{N}_0$ is sufficiently large that the Schwarz kernels of $P$ and $P'$ are continuous away from the side faces of $M_0^2$. Indeed, suppose we have done so. Then using (3.2.10) and

(3.2.9) with $N$ sufficiently large we can decompose $P$, $P'$ and $P \circ P'$ into sums of products of operators for which the homomorphism property holds, to show the proposition for general $m$, $m'$.

So now assume that $m, m' < -N_0$ for a large positive integer $N_0$, implying that the kernels $\beta_0^* K_P$ and $\beta_0^* K_{P'}$ are continuous away from the side faces. We first show that $N_p(P(1-\chi)P') = 0$, where $\chi$ is smooth, supported in a small neighborhood of $p \in M$, and identically 1 near $p$. Let $\varphi : \mathcal{U}' \to \mathcal{U}$ be as before, $(u,v)$ linear coordinates on $T_p^+ M$ with $u$ boundary defining function and $(x,y) = (u,v) \circ \varphi^{-1}$ coordinates near $p$. As explained earlier, we can assume that the coordinates are chosen so that $\det \overline{g}\big|_p = 1$. Also let $(\tilde{\tilde{x}}, \tilde{\tilde{y}})$ be a copy of the coordinate system $(x,y)$ on the right factor of $M^2$. Then $\big(x, y, \hat{t} = \tilde{\tilde{x}}/x, \hat{Y} = (\tilde{\tilde{y}} - y)/x\big)$ are a valid coordinate system in $M_0^2$ away from $B_{10}$. For $f \in C_c^\infty(T_p^+ M)$ and disregarding the densities for convenience we have, for $r > 0$ small and $u > 0$

$$\begin{aligned}
&R_r^* \varphi^* P(1-\chi) P' (\varphi^{-1})^* R_{1/r}^* f(u,w) \\
&= \int \Big( \int K_P\big(r(u,w), \widetilde{z}\big)(1-\chi(\widetilde{z}))K_{P'}\big(\widetilde{z}, r(u\hat{t}, w + u\hat{Y})\big) dV_g(\widetilde{z}) \Big) f(u\hat{t}, w + u\hat{Y}) \frac{|d\hat{t} d\hat{Y}|}{\hat{t}^{n+1}}.
\end{aligned} \tag{3.2.20}$$

In (3.2.20) we evaluated the kernel factors $K_P(z, \widetilde{z})$ and $K_{P'}(\widetilde{z}, \tilde{\tilde{z}})$ at $z = r(u,w)$ and $\tilde{\tilde{z}} = r(u\hat{t}, w + u\hat{Y})$, without writing explicitly the $\varphi$. The fact that $(1-\chi)$ vanishes near $p$ implies that for small $r$ the innermost integrand in (3.2.20) is supported away from the boundary of the triple diagonal in $M^3$. Thus if $\widetilde{\chi} \in C^\infty(M)$ is supported in the zero set of $(1-\chi)$ and is identically 1 near $p$, we have

$$\widetilde{\chi}(z) K_P(z, \widetilde{z})(1-\chi(\widetilde{z}))K_{P'}(\widetilde{z}, \tilde{\tilde{z}})\widetilde{\chi}(\tilde{\tilde{z}}) \in \mathcal{A}_{phg}^{E_{10}, E_{01}+F_{10}, F_{01}}(M^3), \tag{3.2.21}$$

where $E_{10}$, $E_{01} + F_{10}$, $F_{01}$ correspond to expansions at $\partial M \times M^2$, $M \times \partial M \times M$ and $M^2 \times \partial M$ respectively. By the assumption $\mathrm{Re}(E_{01} + F_{10}) > n$, (3.2.21) is integrable in $\widetilde{z}$, and for small enough $r$ the innermost integrand in (3.2.20) can be replaced by (3.2.21) evaluated at $z = r(u,w)$, $\tilde{\tilde{z}} = r(u\hat{t}, w + u\hat{Y})$; the latter has an expansion in $r$ with the exponent of $r$

in the leading order term positive, by the assumption $\mathrm{Re}(E_{10}+F_{01})>0$, so the same holds for the innermost integral, thus (3.2.20) vanishes in the limit as $r\to 0$.

We have established that it suffices to show that $N_p(P\chi P')=N_p(P)N_p(P')$. We will express the action of $P$, $P'$ in coordinates as in (3.2.19). Let $(x,y,\widetilde{x},\widetilde{y})$ and $(\widetilde{x},\widetilde{y},\tilde{\widetilde{x}},\tilde{\widetilde{y}})$ be copies of the same coordinate system near $(p,p)\in M^2$ chosen as before. If $u$ is supported near $p$ in $M$ we have

$$Pu(x,y)=\int \beta_0^*K_P^l\Big(\widetilde{x},\widetilde{y},\frac{x}{\widetilde{x}},\frac{y-\widetilde{y}}{\widetilde{x}}\Big)u(\widetilde{x},\widetilde{y})\frac{|d\widetilde{x}d\widetilde{y}|}{\widetilde{x}^{n+1}} \tag{3.2.22}$$

$$P'u(\widetilde{x},\widetilde{y})=\int \beta_0^*K_{P'}^l\Big(\tilde{\widetilde{x}},\tilde{\widetilde{y}},\frac{\widetilde{x}}{\tilde{\widetilde{x}}},\frac{\widetilde{y}-\tilde{\widetilde{y}}}{\tilde{\widetilde{x}}}\Big)u(\tilde{\widetilde{x}},\tilde{\widetilde{y}})\frac{|d\tilde{\widetilde{x}}d\tilde{\widetilde{y}}|}{\tilde{\widetilde{x}}^{n+1}} \tag{3.2.23}$$

Then with a computation as in (3.2.18) we immediately obtain that for $f\in C_c^\infty(T_p^+M)$

$$N_p(P)f(u,w)=\int \beta_0^*K_P^l\Big(0,0,\frac{u}{\widetilde{x}},\frac{w-\widetilde{y}}{\widetilde{x}}\Big)f(\widetilde{x},\widetilde{y})\frac{|d\widetilde{x}d\widetilde{y}|}{\widetilde{x}^{n+1}}$$

$$N_p(P')f(\widetilde{u},\widetilde{w})=\int \beta_0^*K_{P'}^l\Big(0,0,\frac{\widetilde{u}}{\tilde{\widetilde{x}}},\frac{\widetilde{w}-\tilde{\widetilde{y}}}{\tilde{\widetilde{x}}}\Big)f(\tilde{\widetilde{x}},\tilde{\widetilde{y}})\frac{|d\tilde{\widetilde{x}}d\tilde{\widetilde{y}}|}{\tilde{\widetilde{x}}^{n+1}};$$

We will now compute the model operator of the composition. First write

$$\begin{aligned}&P\chi P'f(x,y)\\&=\int \beta_0^*K_P^l\Big(\widetilde{x},\widetilde{y},\frac{x}{\widetilde{x}},\frac{y-\widetilde{y}}{\widetilde{x}}\Big)\int \chi(\widetilde{x},\widetilde{y})\beta_0^*K_{P'}^l\Big(\tilde{\widetilde{x}},\tilde{\widetilde{y}},\frac{\widetilde{x}}{\tilde{\widetilde{x}}},\frac{\widetilde{y}-\tilde{\widetilde{y}}}{\tilde{\widetilde{x}}}\Big)f(\tilde{\widetilde{x}},\tilde{\widetilde{y}})\frac{|d\tilde{\widetilde{x}}d\tilde{\widetilde{y}}|}{\tilde{\widetilde{x}}^{n+1}}\frac{|d\widetilde{x}d\widetilde{y}|}{\widetilde{x}^{n+1}}.\end{aligned}$$

Upon making a change of variables in each integration to rescale, we find that for small $r$

$$\begin{aligned}&R_r^*\varphi^*P\chi P'(\varphi^{-1})^*R_{1/r}^*f(u,w)\\&=\int\int \beta_0^*K_P^l\Big(r\widetilde{x},r\widetilde{y},\frac{u}{\widetilde{x}},\frac{w-\widetilde{y}}{\widetilde{x}}\Big)\chi(r\widetilde{x},r\widetilde{y})\beta_0^*K_{P'}^l\Big(r\tilde{\widetilde{x}},r\tilde{\widetilde{y}},\frac{\widetilde{x}}{\tilde{\widetilde{x}}},\frac{\widetilde{y}-\tilde{\widetilde{y}}}{\tilde{\widetilde{x}}}\Big)f(\tilde{\widetilde{x}},\tilde{\widetilde{y}})\frac{|d\tilde{\widetilde{x}}d\tilde{\widetilde{y}}|}{\tilde{\widetilde{x}}^{n+1}}\frac{|d\widetilde{x}d\widetilde{y}|}{\widetilde{x}^{n+1}}.\end{aligned} \tag{3.2.24}$$

Note that the integrand is $L^1$. Indeed, by our assumption on $m,m'$ both kernels are continuous away from the side faces of $M_0^2$ and the integration in $(\tilde{\widetilde{x}},\tilde{\widetilde{y}})$ is over a compact subset of the open upper half plane. Moreover, since $\chi$ is supported in a small neighborhood of $p$, $\widetilde{x}$ is bounded in its support and thus one only needs to be careful about the behavior of the integrand as $\widetilde{x} \to 0$. Since $\widetilde{x}/x$ is a defining function for $B_{01}$, $\beta_0^* K_P^l\Big(r\widetilde{x}, r\widetilde{y}, \frac{u}{\tilde{\widetilde{x}}}, \frac{w-\widetilde{y}}{\widetilde{x}}\Big) = \beta_0^* K_P^l\Big(r\widetilde{x}, r\widetilde{y}, \frac{u}{\tilde{\widetilde{x}}}, \frac{u}{\widetilde{x}}\frac{w-\widetilde{y}}{u}\Big)$ has a polyhomogeneous expansion in $u/\widetilde{x}$ with index set $-E_{01}$ as $\widetilde{x} \to 0$, On the other hand, $\beta_0^* K_{P'}$ has an expansion with index set $F_{10}$ in $\widetilde{x}/\tilde{\widetilde{x}}$ and therefore the integrand in (3.2.24) is integrable by the assumption $\mathrm{Re}(E_{01} + F_{10}) > n$. Now by dominated convergence we can take the limit as $r \to 0$ to find that

$$\begin{aligned} &N_p(P\chi P')(f)(u,w) \\ &\quad = \int\int \beta_0^* K_P^l\Big(0, 0, \frac{u}{\widetilde{x}}, \frac{w-\widetilde{y}}{\widetilde{x}}\Big)\beta_0^* K_{P'}^l\Big(0, 0, \frac{\widetilde{x}}{\tilde{\widetilde{x}}}, \frac{\widetilde{y}-\tilde{\widetilde{y}}}{\tilde{\widetilde{x}}}\Big) f(\tilde{\widetilde{x}}, \tilde{\widetilde{y}}) \frac{|d\tilde{\widetilde{x}}d\tilde{\widetilde{y}}|}{\tilde{\widetilde{x}}^{n+1}} \frac{|d\widetilde{x}d\widetilde{y}|}{\widetilde{x}^{n+1}}, \end{aligned}$$

which is the same expression that one finds upon composing (3.2.22) and (3.2.23). $\square$

### 3.3 The Pseudodifferential Property

In this section show that the normal operator $\mathcal{N}_g$ is a 0-pseudodifferential operator, namely that $\mathcal{N}_g \in \Psi_0^{-1,n,n}(M)$, and study the distance function induced by $g$ as an intermediate step. Once the pseudodifferential property of $\mathcal{N}$ has been established, we use it to extend $I$ to larger weighted $L^2$ spaces than those of Section 3.1.

We first state and prove a technical lemma, which is similar to Proposition 19 in [CH16].

**Lemma 3.3.1.** *Let $(\mathring{M}, g)$ be a simple AH manifold. The map*

$$\begin{aligned} \Phi : T^*\mathring{M} &\to M_0^2 \\ (z,\xi) &\mapsto \big(z, \exp_z(\xi^\#)\big) \end{aligned}$$

*extends smoothly to a map $\widetilde{\Phi} : {}^0T^*M \to M_0^2$, where we are using the canonical identification of ${}^0T^*M\big|_{\mathring{M}} = \big({}^0T^*M\big)^\circ$ and $T^*\mathring{M}$. Here $\#$ raises an index with respect to the metric $g$. Moreover, the differential of $\widetilde{\Phi}$ at $(z,0) \in {}^0T^*M\big|_{\partial M}$ has full rank.*

*Proof.* We will rewrite the map $\Phi$ as a composition of maps. Consider the bundle ${}^0T^*M \times {}^0T^*M =: ({}^0T^*M)^2$ over $M^2$. Following [CH16], we let ${}^\Phi T^*M_0^2 := \beta_0^*\big(({}^0T^*M)^2\big)$; it is a bundle over $M_0^2$ with bundle projection denoted by ${}^{\Phi}\pi$. Define $\psi : {}^0T^*M \to {}^{\Phi}T^*M_0^2$ by

$$\psi(z,\xi) = \begin{cases} (z,-\xi,z,\xi\,), & \text{if } (z,\xi\,) \in {}^0T^*M\big|_{\mathring{M}} \\ \big(e_z, \big(z,-\xi,z,\xi\,\big)\big)\,, & \text{if } (z,\xi\,) \in {}^0T^*M\big|_{\partial M}, \end{cases} \tag{3.3.1}$$

where $e_z$ denotes the canonical origin in the fiber of the front face over the point $(z,z)$ [5]. Using coordinates $(x,y,\widetilde{x},\widetilde{y})$ near a fixed point $(p,p) \in \partial\Delta\iota \subset M^2$ and projective coordinates $\big(x,y,t=\widetilde{x}/x, Y=(\widetilde{y}-y)/x\big)$ in $M_0^2$ away from $B_{10}$, in terms of which $\Delta\iota_0 = \{t=1, Y=0\}$ and $B_{11} = \{x=0\}$, one can see that $\psi$ is a smooth map. Indeed, recalling that ${}^{\Phi}T^*M_0^2 \subset M_0^2 \times ({}^0T^*M)^2$ and suppressing the base point of 0-covectors for $(z,\xi) \in {}^0T^*M\big|_{\mathring{M}}$, we can write in terms of the projective coordinates $\psi(z,\xi\,) = \big((x,y,1,0),(-\xi,\xi\,)\big)$, which extends smoothly down to $x=0$.

We will next compose $\psi$ with the flow of an appropriately chosen vector field on ${}^{\Phi}T^*M_0^2$. In [CH16], the authors analyze the Hamiltonian vector field $X$ associated to the metric Lagrangian $\mathcal{L}_g = |\xi|_g^2/2$ on $T^*\mathring{M}$ viewed as a vector field on ${}^0T^*M$ , under the identification $T^*\mathring{M} \leftrightarrow {}^0T^*M\big|_{\mathring{M}}$; it turns out that $X$ is smooth on ${}^0T^*M$. Let $(x,y)$ be coordinates near $p$ such that the metric is written in normal form $g = \dfrac{dx^2 + (h_x)_{\sigma\tau}dy^\sigma dy^\tau}{x^2}$ as in (3.1.1) and also let $(z,\xi) = (x,y,\widetilde{\zeta},\widetilde{\eta})$ be coordinates for ${}^0T^*M$ near the fiber over $p$ such that $\xi = \widetilde{\zeta}x^{-1}dx + \widetilde{\eta}_\alpha x^{-1}dy^\alpha$. Then in terms of those coordinates

$$X = x\widetilde{\zeta}\partial_x + xh_x^{\sigma\tau}\widetilde{\eta}_\tau\partial_{y^\sigma} - \left(h_x^{\sigma\tau} + \frac{1}{2}x\partial_x h_x^{\sigma\tau}\right)\widetilde{\eta}_\sigma\widetilde{\eta}_\tau\partial_{\widetilde{\zeta}} + \left(\widetilde{\zeta}\widetilde{\eta}_\sigma - \frac{1}{2}x\partial_{y^\sigma}h_x^{\tau\lambda}\widetilde{\eta}_\tau\widetilde{\eta}_\lambda\right)\partial_{\widetilde{\eta}_\sigma}, \tag{3.3.2}$$

which shows that $X$ is smooth down to $\partial\,{}^0T^*M$, and tangent to it. Thus if $X_R = (0,X)$ denotes the Hamiltonian vector field on the right factor of $({}^0T^*M)^2$, we can also deduce that $X_R$ extends smoothly to the boundary faces of $({}^0T^*M)^2$, meeting them tangentially. Moreover, as shown in [CH16], $X_R$ lifts from the interior of $({}^0T^*M)^2$ via $\beta_0\big|_{(M_0^2)^\circ}$ to a vector field on $({}^{\Phi}T^*M_0^2)^\circ$ (still denoted $X_R$), which extends smoothly to its boundary faces and is tangent to all of them. More strongly, $X_R$ is tangent to the level sets of $\mathcal{L}_g = \frac{1}{2}\big(\widetilde{\zeta}^{\,2} + h^{\sigma\tau}\widetilde{\eta}_\sigma\widetilde{\eta}_\tau\big)$

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[5] Only the the right 0-covector in the definition of (3.3.1) will enter the subsequent arguments; nothing essential changes if one choses to define, for instance, $\psi(z,\xi) = (z,0,z,\xi)$ over the interior of ${}^0T^*M$ and accordingly at the boundary.

on each factor of $({}^0T^*M)^2$. Those level sets extend smoothly to the boundary, both in $({}^0T^*M)^2$ and in ${}^\Phi T^*M_0^2$, resulting in smooth compact manifolds with corners in both cases. Thus the flow of $X_R$ in ${}^\Phi T^*M_0^2$ is complete. Now let

$$\begin{aligned}\widetilde{\Phi} : {}^0T^*M &\to M_0^2 \\ \big(z,\xi\big) &\mapsto {}^\Phi\pi \circ \widetilde{\varphi}_1 \circ \psi\big(z,\xi\big),\end{aligned}$$

where $\widetilde{\varphi}_t$ is the flow of $X_R$. This map is smooth as a composition of smooth maps and it extends $\Phi$. Moreover, it maps $0 \in {}^0T_p^*M$ at $p \in \partial M$ to $\partial\Delta\iota_0\big|_{(p,p)} = e_p$, since from (3.3.2) it follows that $X_R$ vanishes at $\psi(p,0) = (e_p,(0,0)) \in {}^\Phi T^*M_0^2$.

It remains to show that $d\widetilde{\Phi}$ has full rank in a sufficiently small neighborhood of $(p,0) \in {}^0T^*M\big|_{\partial M}$. In ${}^0T^*\overline{M}\big|_{\mathring{M}}$ we write $\widetilde{\Phi}(z,\xi) = \exp_z(\xi^\#)$ and in coordinates $(x,y,t,Y)$ as before

$$\begin{aligned}(x,y,&t,Y) = \widetilde{\Phi}(x,y,\widetilde{\zeta},\widetilde{\eta}) \\ &= \left(x, y, \frac{\widetilde{x}\circ\big(\exp_{(x,y)}(\widetilde{\zeta}x\partial_x + \widetilde{\eta}_\sigma h^{\sigma\tau}x\partial_{y^\tau})\big)}{x}, \frac{\widetilde{y}\circ\big(\exp_{(x,y)}(\widetilde{\zeta}x\partial_x + \widetilde{\eta}_\sigma h^{\sigma\tau}x\partial_{y^\tau})\big) - y}{x}\right).\end{aligned} \tag{3.3.3}$$

For $x > 0$ we can compute its Jacobian matrix in these coordinates at the 0-covector, using the fact that the differential of the exponential map at 0 is the identity. We find

$$d\widetilde{\Phi}_{(x,y,0,0)} = \begin{pmatrix} \mathrm{Id} & 0 \\ * & \overline{g}^{-1}_{(x,y)} \end{pmatrix}, \tag{3.3.4}$$

where $\overline{g}^{-1}_{(x,y)}$ is the matrix of the dual metric corresponding to $\overline{g} = x^2 g$ in $(x,y)$ coordinates for $x > 0$. Since we have already established that $\widetilde{\Phi}$ is smooth in a neighborhood of $e_p$, (3.3.4) also holds down to $x = 0$ and this completes the proof. $\square$

The behavior of the distance function on AH manifolds away from the diagonal has been studied by various authors, see for instance [SW16], [CH16] and [GGS$^+$], and also [MSV14] for small perturbations of hyperbolic metric. As Proposition 3.3.2 below indicates, provided $(\mathring{M}, g)$ is simple, the lift of the distance function to $M_0^2$ is smooth away from $\Delta\iota_0$ and the side faces, however our analysis of $\mathcal{N}_g$ will also require smoothness of the lift of its square in a neighborhood of $\Delta\iota_0$, all the way to the front face. We are not aware of this fact explicitly stated in the literature, so we provide a proof.

**Proposition 3.3.2.** *Let $(\mathring{M}, g)$ be a simple AH manifold and let $\rho : \mathring{M}^2 \to \mathbb{R}$ be the geodesic distance function. There exists $\alpha \in C^\infty(M_0^2 \backslash \Delta\iota_0)$ such that*

$$\beta_0^* \rho = \alpha - \log(x_{10}) - \log(x_{01}),$$

*where $x_{10}$ and $x_{01}$ are defining functions for the left and right face of $M_0^2$ respectively. Moreover, $\beta_0^*\rho^2$ extends to a smooth function on $M_0^2 \backslash (B_{10} \cup B_{01})$.*

*Proof.* The first statement follows from work in [SW16], [CH16] and [GGS+] (see [GGS+, Remark 7]). We show the second statement. Assume without loss of generality that $x_{10},\ x_{01} \equiv 1$ in a neighborhood of $\Delta\iota_0$. Since $\rho^2$ is smooth near $\Delta\iota \cap \mathring{M}^2$ and thus $\beta_0^*\rho^2$ extends to a function in $C^\infty(M_0^2 \backslash (\partial\Delta\iota_0 \cup B_{10} \cup B_{01}))$, it is enough to show that $\beta_0^*\rho^2$ extends to be smooth in a neighborhood of $\partial\Delta\iota_0$. By the Inverse Function Theorem, Lemma 3.3.1 implies that $\widetilde{\Phi}$ restricted to a neighborhood of a point $(p, 0) \in {}^0T^*M\big|_{\partial M}$ is invertible. The inverse, defined in a neighborhood $U \subset M_0^2$ of $\partial\Delta\iota_0\big|_{(p,p)}$, is smooth all the way to the front face. In $U \cap (M_0^2)^\circ$

$$\beta_0^*\rho^2(z, \widetilde{z}) = |\exp_z^{-1}(\widetilde{z})|_g^2 = \left|\left(\widetilde{\Phi}^{-1}(z, \widetilde{z})\right)^{\#}\right|_g^2 = \left|\widetilde{\Phi}^{-1}(z, \widetilde{z})\right|_{g^{-1}}^2 \tag{3.3.5}$$

using the identification $(M_0^2)^\circ \leftrightarrow \mathring{M}^2$. Since $g$ induces a non-degenerate quadratic form on the fibers of ${}^0T^*M$, smooth all the way to the boundary, (3.3.5) extends smoothly to $\partial\Delta\iota_0$ and this finishes the proof. □

The proof of the following lemma is essentially contained in [SU04]. Recall that by simplicity of $(\mathring{M}, g)$ the exponential map at any point is a diffeomorphism onto $\mathring{M}$ and thus it can be used to define a global coordinate system on $\mathring{M}$.

**Lemma 3.3.3.** *Let $M$ be a simple AH manifold and let $z = (z^0, \ldots, z^n)$ and $\widetilde{z} = (\widetilde{z}^0, \ldots, \widetilde{z}^n)$ two copies of the same global coordinate system in each of the two factors of $\mathring{M}^2$. The kernel* *of* $\mathcal{N}_g$*, viewed as a section of* $\Omega_0^{1/2}(M^2)$*, is given by* $K_{\mathcal{N}_g}(z,\widetilde{z})\cdot\gamma_0(z)\otimes\gamma_0(\widetilde{z})$*, where*

$$K_{\mathcal{N}_g}(z,\widetilde{z}) = \frac{2|\det(\partial_{z\widetilde{z}}\rho^2/2)|}{\rho^n(z,\widetilde{z})\sqrt{\det g(z)}\sqrt{\det g(\widetilde{z})}}. \tag{3.3.6}$$

*Proof.* Let $f\in\dot{C}^\infty(M)$. We compute $\mathcal{N}_g f$, viewed as a function:

$$\begin{aligned}\mathcal{N}_g f(z) &= \int_{S_z^*\mathring{M}}\int_{-\infty}^{\infty} f\big(\exp_z(t\xi^\#)\big)dt d\mu_g(\xi) = 2\int_{S_z^*\mathring{M}}\int_0^\infty f\big(\exp_z(t\xi^\#)\big)dt d\mu_g(\xi)\\ &= 2\int_{T_z^*\mathring{M}} f\big(\exp_z(\xi^\#)\big)|\xi|_g^{-n}\sqrt{\det g^{-1}(z)}d\xi = 2\int_{\mathring{M}}\frac{f(\widetilde{z})|\det\big(d_{\widetilde{z}}\exp_z^{-1}(\widetilde{z})^\flat\big)|}{|\exp_z^{-1}(\widetilde{z})|_g^n\sqrt{\det g(z)}}d\widetilde{z}\end{aligned}$$

where the third equality follows by polar coordinates in the inner product space $(T_z^*\mathring{M}, g^{-1}(z))$. Now $|\exp_z^{-1}(\widetilde{z})|_g = \rho(z,\widetilde{z})$ and by the Gauss Lemma $\exp_z^{-1}(\widetilde{z})^\flat = -\rho d_z\rho$, so

$$\mathcal{N}_g f(z) = \int_{\mathring{M}}\frac{2|\det(\partial_{z\widetilde{z}}\rho^2/2)|f(\widetilde{z})}{\rho^n(z,\widetilde{z})\sqrt{\det g(z)}\sqrt{\det g(\widetilde{z})}}dV_g(\widetilde{z}).$$

Multiplying both sides by the half density $\gamma_0(z)$ and viewing $\mathcal{N}_g$ as an operator acting on the half density $f\cdot\gamma_0(\widetilde{z})$ we have the claim. □

We now prove the following key proposition:

**Proposition 3.3.4.** *Let* $(\mathring{M}^{n+1}, g)$ *be a simple AH manifold. Then* $\mathcal{N}_g\in\Psi_0^{-1,n,n}(M)$*. Moreover, it is elliptic.*

*Proof.* We examine the Schwartz kernel of $\mathcal{N}_g$ on $\mathring{M}^2$ and on the stretched product $M_0^2$. As noted in [SU04], the form of the kernel in Lemma 3.3.3 implies that in open subsets of $\mathring{M}^2$ the kernel of $\mathcal{N}_g$ agrees with the kernel of a pseudodifferential operator of order $-1$ with principal symbol $C_n|\xi|_g^{-1}$. Since smooth sections of $\big(\Omega_0^{1/2}(M^2)\big)$ lift to smooth sections of $\Omega_0^{1/2}(M_0^2)$ it suffices to study the behavior of $K_{\mathcal{N}_g}(z,\widetilde{z})$ in (3.3.6) and its pullback to $M_0^2$ as $z$, $\widetilde{z}\to\partial M$, both away from, and near the diagonal. Throughout the proof, $z=(x,y)$, $\widetilde{z}=(\widetilde{x},\widetilde{y})$ are representations in terms of two copies of the same coordinate system in each factor of $\mathring{M}^2$ such that $x$, $\widetilde{x}$ are boundary defining functions.

First note that $|\det(\partial_{z\widetilde{z}}\rho^2/2)| = |\det\big(d_{\widetilde{z}}\exp_z^{-1}(\widetilde{z})^\flat\big)|$ by the proof of Lemma 3.3.3, so by simplicity $\det(\partial_{z\widetilde{z}}\rho^2/2)\neq 0$ on $\mathring{M}^2$ and the absolute value can be ignored in the process of examining the smoothness properties of $K_{\mathcal{N}_g}$ and $\beta_0^* K_{\mathcal{N}_g}$. Moreover, we observe a simplification of (3.3.6) away from the diagonal. We have that $\partial^2_{z\widetilde{z}}(\rho^2/2) = \rho\partial^2_{z\widetilde{z}}\rho + \partial_z\rho \otimes \partial_{\widetilde{z}}\rho$. Since for $H \in \mathbb{R}^{d\times d}$ and $u, v \in \mathbb{R}^d$ one has $\det(H + u\otimes v) = \det(H) + (\mathrm{adj}(H)u)\cdot v$ by the matrix determinant lemma, where $\mathrm{adj}(H)$ is the adjugate matrix of $H$ and $\cdot$ denotes the Euclidean dot product, we have

$$\det\left(\partial^2_{z\widetilde{z}}(\rho^2/2)\right) = \rho^{n+1}\det(\partial^2_{z\widetilde{z}}\rho) + \rho^n\left(\mathrm{adj}(\partial^2_{z\widetilde{z}}\rho)\partial_z\rho\right)\cdot\partial_{\widetilde{z}}\rho.$$

Observe that the first term vanishes away from the diagonal. Indeed, if $z \neq \widetilde{z}$ the Gauss Lemma yields $|d_z\rho(z,\widetilde{z})|_g = 1$, thus the rank of the map $d_z\rho(z,\cdot) : \mathring{M}\backslash\{z\} \to S^*_z\mathring{M}$ is at most $n$. Therefore, $\det(\partial^2_{\widetilde{z}z}\rho) = 0$ and thus away from the diagonal we have

$$K_{\mathcal{N}_g}(z,\widetilde{z}) = \frac{2|\left(\mathrm{adj}(\partial^2_{z\widetilde{z}}\rho)\partial_z\rho\right)\cdot\partial_{\widetilde{z}}\rho|}{\sqrt{\det g(z)}\sqrt{\det g(\widetilde{z})}}. \tag{3.3.7}$$

We first examine $K_{\mathcal{N}_g}(z,\widetilde{z})$ on $\mathring{M}^2$ away from the diagonal when $z \to \partial M$ or $\widetilde{z} \to \partial M$. By Proposition 3.3.2, if , for $z, \widetilde{z}$ away from the diagonal we have

$$\rho(z,\widetilde{z}) = \alpha(x,y,\widetilde{x},\widetilde{y}) - \log(x) - \log(\widetilde{x}),$$

where $\alpha \in C^\infty\left(M^2\backslash\Delta\iota\right)$. Since $\partial^2_{z\widetilde{z}}\rho = \partial^2_{z\widetilde{z}}\alpha$, $\mathrm{adj}(\partial^2_{z\widetilde{z}}\rho) \in C^\infty(M^2\backslash\Delta\iota)$. Moreover, $\sqrt{\det g(z)} = x^{-n-1}\sqrt{\det\overline{g}(z)}$ and $\sqrt{\det g(\widetilde{z})} = \widetilde{x}^{\,-n-1}\sqrt{\det\overline{g}(\widetilde{z})}$ with $\det\overline{g}(z), \det\overline{g}(\widetilde{z}) \in C^\infty(M)$ and non-vanishing. Finally, $\partial_z\rho \in x^{-1}C^\infty(M^2\setminus\Delta\iota)$ and similarly for $\partial_{\widetilde{z}}\rho$, thus

$$K_{\mathcal{N}_g}(z,\widetilde{z}) \in x^n\widetilde{x}^{\,n}C^\infty\left(M^2\backslash\Delta\iota\right).$$

Now we have to examine the pullback $\beta_0^*K_{\mathcal{N}_g}$ of $K_{\mathcal{N}_g}$ to the stretched product $M_0^2$. The coordinate systems $(x,y)$ and $(\widetilde{x},\widetilde{y})$ we used before induce coordinate systems in various neighborhoods of $M_0^2$. First let $\mathcal{U}$ be a neighborhood of $B_{11}\backslash\partial\Delta\iota_0$, disjoint from the diagonal and $B_{10}$. On $\mathcal{U}$ we use projective coordinates

$$x, \quad y, \quad t = \frac{\widetilde{x}}{x}, \quad Y = \frac{\widetilde{y} - y}{x}, \tag{3.3.8}$$

in terms of which $t$ is a defining function for $B_{01}$ and $x$ is a defining function for $B_{11}$. By Proposition 3.3.2, in $\mathcal{U}$ we have $\beta_0^*\rho = \widetilde{\alpha} - \log(t)$, $\widetilde{\alpha} \in C^\infty(\mathcal{U})$. Thus the chain rule yields

$$\beta_0^*(\partial_x\rho, \partial_y\rho) = \left(\partial_x\widetilde{\alpha} - tx^{-1}\left(\partial_t\widetilde{\alpha} - t^{-1}\right) - x^{-1}Y^\sigma\partial_{Y^\sigma}\widetilde{\alpha},\ \partial_y\widetilde{\alpha} - x^{-1}\partial_Y\widetilde{\alpha}\right) = x^{-1}\varphi,$$
$$\beta_0^*(\partial_{\widetilde{x}}\rho, \partial_{\widetilde{y}}\rho) = \left(x^{-1}\left(\partial_t\widetilde{\alpha} - t^{-1}\right),\ x^{-1}\partial_Y\widetilde{\alpha}\right) = t^{-1}x^{-1}\varphi',$$

where $\varphi$, $\varphi'$ have components in $C^\infty(\mathcal{U})$, and further

$$\begin{aligned}
\beta_0^*\partial^2_{x\widetilde{x}}\rho =& x^{-2}\left(-\partial_t\widetilde{\alpha} + x\partial^2_{xt}\widetilde{\alpha} - t\partial^2_{tt}\widetilde{\alpha} - Y^\lambda\partial^2_{Y^\lambda t}\widetilde{\alpha}\right) = x^{-2}\psi_{00},\\
\beta_0^*\partial^2_{y^\sigma\widetilde{x}}\rho =& x^{-2}\left(x\partial^2_{y^\sigma t}\widetilde{\alpha} - \partial^2_{Y^\sigma t}\widetilde{\alpha}\right) = x^{-2}\psi_{\sigma 0},\\
\beta_0^*\partial^2_{y^\sigma\widetilde{y}^\tau}\rho =& x^{-2}\left(x\partial^2_{y^\sigma Y^\tau}\widetilde{\alpha} - \partial^2_{Y^\sigma Y^\tau}\widetilde{\alpha}\right) = x^{-2}\psi_{\sigma\tau},\\
\beta_0^*\partial^2_{x\widetilde{y}^\tau}\rho =& x^{-2}\left(-\partial_{Y^\tau}\widetilde{\alpha} + x\partial^2_{xY^\tau}\widetilde{\alpha} - t\partial^2_{tY^\tau}\widetilde{\alpha} - Y^\lambda\partial^2_{Y^\lambda Y^\tau}\widetilde{\alpha}\right) = x^{-2}\psi_{0\tau},
\end{aligned} \tag{3.3.9}$$

where $\psi_{ij} \in C^\infty(\mathcal{U})$. Note that $\partial_x$, $\partial_y$ have different meanings in the left and right hand sides of the above equations. Since for $H \in \mathbb{R}^{d\times d}$ and $\lambda \in \mathbb{R}$ we have $\operatorname{adj}(\lambda H) = \lambda^{d-1}\operatorname{adj}(H)$ we find $\beta_0^*(\operatorname{adj}(\partial^2_{z\widetilde{z}}\rho)) \in x^{-2n}C^\infty(\mathcal{U};\mathbb{R}^{(n+1)\times(n+1)})$. On the other hand, $\beta_0^*\sqrt{\det g(z)} = x^{-n-1}\widetilde{g}_1$ and $\beta_0^*\sqrt{\det g(\widetilde{z})} = t^{-n-1}x^{-n-1}\widetilde{g}_2$ with $\widetilde{g}_j \in C^\infty(\mathcal{U})$ and non-vanishing for $j = 1, 2$. By (3.3.7) we conclude that $\beta_0^*K_{\mathcal{N}_g} \in t^nC^\infty(\mathcal{U})$. This shows that $\beta_0^*K_{\mathcal{N}_g}$ has the claimed behavior away from $B_{10}$ and $\Delta\iota_0$; moreover, the fact that (3.3.6) is symmetric implies that this is also true away from $B_{01}$ and $\Delta\iota_0$.

We now examine $\beta_0^*K_{\mathcal{N}_g}$ in a neighborhood $\mathcal{W}$ of a point in $B_{10}\cap B_{11}\cap B_{01}$ away from $\Delta\iota_0$. Near such a point we have $|y - \widetilde{y}| \neq 0$, hence at least one of the functions $y^\sigma - \widetilde{y}^\sigma$ does not vanish. We may assume without loss of generality that $y^n - \widetilde{y}^n > 0$ and use coordinates

$$r = y^n - \widetilde{y}^n, \quad \theta = \frac{x}{r}, \quad \widetilde{\theta} = \frac{\widetilde{x}}{r}, \quad \widehat{Y}^{\hat{\lambda}} = \frac{y^{\hat{\lambda}} - \widetilde{y}^{\hat{\lambda}}}{r}, \quad y, \tag{3.3.10}$$

where $\hat{\lambda} = 1, \dots, n-1$. A computation using the chain rule yields

$$\beta_0^*\partial_x = r^{-1}\partial_\theta, \quad \beta_0^*\partial_{y^\tau} = r^{-1}\big((r\partial_r - \widetilde{\theta}\partial_{\widetilde{\theta}})\delta^n_\tau + V_\tau\big), \quad V_\tau \in \mathcal{V}_b(M_0^2),\ d\widetilde{\theta}(V_\tau) = dr(V_\tau) = 0$$
$$\beta_0^*\partial_{\widetilde{x}} = r^{-1}\partial_{\widetilde{\theta}}, \quad \beta_0^*\partial_{\widetilde{y}^\tau} = r^{-1}\big(-(r\partial_r - \theta\partial_\theta)\delta^n_\tau + \widetilde{V}_\tau\big), \quad \widetilde{V}_\tau \in \mathcal{V}_b(M_0^2),\ d\theta(\widetilde{V}_\tau) = dr(\widetilde{V}_\tau) = 0.$$

In terms of (3.3.10) $\beta_0^*\rho = \widetilde{a} - \log(\theta) - \log(\widetilde{\theta}),\ \widetilde{a} \in C^\infty(\mathcal{W})$, so

$$\beta_0^*(\partial_x\rho, \partial_y\rho) = (-(r\theta)^{-1}, 0) + r^{-1}\breve{\varphi}, \qquad \beta_0^*(\partial_{\widetilde{x}}\rho, \partial_{\widetilde{y}}\rho) = (-(r\widetilde{\theta})^{-1}, 0) + r^{-1}\breve{\varphi}\,', \quad (3.3.11)$$

where $\breve{\varphi}, \breve{\varphi}\,' \in C^\infty(\mathcal{W}; \mathbb{R}^{n+1})$. Moreover, since $(r\partial_r - \theta\partial_\theta)((r\theta)^{-1}) = 0$ and $(r\partial_r - \widetilde{\theta}\partial_{\widetilde{\theta}})((r\widetilde{\theta})^{-1}) = 0$ we find that $\beta_0^*\partial_{z\widetilde{z}}\,\rho \in r^{-2}C^\infty(\mathcal{W}; \mathbb{R}^{(n+1)\times(n+1)})$. Thus $\beta_0^*\operatorname{adj}(\partial^2_{z\widetilde{z}}\rho) \in r^{-2n}C^\infty(\mathcal{W}; \mathbb{R}^{(n+1)\times(n+1)})$. Noting that $\beta_0^*\sqrt{\det g(z)} = r^{-n-1}\theta^{-n-1}\breve{g}_1$ and $\beta_0^*\sqrt{\det g(\widetilde{z})} = r^{-n-1}\widetilde{\theta}^{-n-1}\breve{g}_2$ with $\breve{g}_j \in C^\infty(\mathcal{W})$ and non-vanishing, we use (3.3.7) again and (3.3.11) to find that $\beta_0^*K_{\mathcal{N}_g} \in \theta^n\widetilde{\theta}^{\,n}C^\infty(\mathcal{W})$. We conclude that $\beta_0^*K_{\mathcal{N}_g} \in C^\infty(M_0^2 \setminus \Delta\iota_0)$ and vanishes to order $n$ on $B_{10}$ and $B_{01}$.

To finish the proof it remains to examine the behavior of the pullback of (3.3.6) near $\partial\Delta\iota_0$. First note that $\beta_0^*\rho\big|_{(M_0^2)^\circ}$ vanishes exactly on $\Delta\iota_0 \cap (M_0^2)^\circ$. By Proposition 3.3.2, $\beta_0^*\rho^2$ is smooth in a neighborhood of $\Delta\iota$, hence it also vanishes on $\partial\Delta\iota_0$. Finally, by Proposition 24 in [CH16], for every $p \in \partial M$, $\beta_0^*\rho$ restricts to a function on $B_{11}\big|_p$, which for each $q \in B_{11}\big|_p$ agrees with the distance between $e_p$ and $q$ induced by the hyperbolic metric $h_p^r$ on $B_{11}\big|_p$. Therefore, $\beta_0^*\rho$ does not vanish on the front face at any point other than $e_p$. We conclude that $\beta_0^*\rho^2 \in C^\infty\big(M_0^2 \setminus (B_{10} \cup B_{01})\big)$ and vanishes exactly on $\Delta\iota_0$.

We now use a variant of the coordinates given by (3.3.8): near $\Delta\iota_0$ and away from $B_{10}$ we use $(z, Z) = (z, (\widetilde{z} - z)/x) = (x, y, t-1, Y)$. Note that in terms of those coordinates $x$ is again a defining function for $B_{11}$ and $\Delta\iota_0$ is expressed as $\{Z = 0\}$. Observe that on $(M_0^2)^\circ$ one has

$$\begin{aligned}\beta_0^*\rho^2\big|_{Z=0} = \beta_0^*\big(\rho^2\big|_{\{z=\widetilde{z}\}}\big) = 0, \quad \partial_{Z^j}(\beta_0^*\rho^2)\big|_{Z=0} = x\beta_0^*\left(\partial_{\widetilde{z}^j}(\rho^2)\big|_{\{z=\widetilde{z}\}}\right) = 0,\\ \partial_{Z^iZ^j}(\beta_0^*\rho^2)\big|_{Z=0} = x^2\beta_0^*\left(\partial_{\widetilde{z}^i\widetilde{z}^j}(\rho^2)\big|_{\{z=\widetilde{z}\}}\right) = 2x^2 g_{ij}(z) = 2\overline{g}_{ij}(z). \qquad (3.3.12)\end{aligned}$$

By smoothness of $\beta_0^*\rho^2$ near $\Delta\iota_0$ we conclude that (3.3.12) holds all the way to the front face. Thus by Taylor's Theorem, viewing $z$ as parameters, we write

$$\beta_0^*\rho^2 = \overline{g}_{ij}(z)Z^iZ^j + b_{klm}(z, Z)Z^kZ^lZ^m$$

where $b_{klm}(z, Z)$ is smooth.

We will now show that the expression $\hat{K}(z,\widetilde{z}) := \frac{2|\det(\partial_{z\widetilde{z}}\rho^2/2)|}{\sqrt{\det g(z)}\sqrt{\det g(\widetilde{z})}}$ pulls back to a non-vanishing smooth function in a neighborhood of the lifted diagonal all the way to the front face. First one can perform computations very analogous to (3.3.9), with $\rho$ replaced by $\rho^2$, to conclude that $x^2\beta_0^*\partial^2_{z\widetilde{z}}(\rho^2/2)$ is a smooth matrix valued function in a neighborhood of $\partial\Delta\iota_0$ (note here that its behavior near $t=0$ is irrelevant for this computation). Also, for $z \in \mathring{M}$ one has $\big|\det(\partial_{z\widetilde{z}}\rho^2/2)|_{z=\widetilde{z}}\big| = |\det g(z)|$. Therefore, since $x^{2n+2}\beta_0^*\big(\sqrt{\det g(z)}\sqrt{\det g(\widetilde{z})}\,\big)$ is smooth and non-vanishing near $\Delta\iota_0$, $\beta_0^*\hat{K}$ is smooth in a neighborhood of the lifted diagonal. Moreover, $\hat{K}\big|_{\Delta\iota\cap\mathring{M}^2} \equiv 2$ implies that $\beta_0^*\hat{K}\big|_{\Delta\iota_0} \equiv 2$.

Now the fact that $\beta_0^*K_{\mathcal{N}_g} \in I^{-1}(M_0^2,\Delta\iota_0)$ follows from a standard argument, which we will outline. Consider $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ with $\varphi \equiv 1$ near 0 and write

$$K_\varphi(z,Z) := K_{\mathcal{N}_g}(z, z-xZ)\varphi(Z) = \frac{a(z,Z)\varphi(Z)}{(\overline{g}_{ij}(z)Z^iZ^j + b_{klm}(z,Z)Z^kZ^lZ^m)^{n/2}}, \tag{3.3.13}$$

where $a$ is smooth with $a(z,0)=2$. Now $K_\varphi$ is smooth in $z,Z$ away from $Z=0$ down to $x=0$ (so the existence of this boundary $x=0$ can be ignored). Moreover, $K_\varphi$ is compactly supported in $Z$ and integrable in $Z$. Writing $r=|Z|_{\overline{g}}$ and $\hat{Z}=r^{-1}Z$ we have $K_\varphi(z,Z) = r^{-n}\widetilde{a}(z,r,\hat{Z})$, where $\widetilde{a}$ is $C^\infty$ in its entries, hence by Taylor's theorem there exists an expansion

$$K_\varphi(z,Z) \sim \sum_{\ell\geq 0}\widetilde{a}_\ell(z,\hat{Z})|Z|_{\overline{g}}^{-n+\ell}, \quad \widetilde{a}_\ell \in C^\infty,\ \widetilde{a}_0 = 2. \tag{3.3.14}$$

This is exactly the setup of Proposition 2.8, Chapter 7 in [Tay11], which implies that $K_\varphi(z,Z) = \int e^{iZ\cdot\widetilde{\xi}}p(z,\widetilde{\xi})d\widetilde{\xi}$, where $p(z,\widetilde{\xi}) \in S_{cl}^{-1}(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$, i.e. $p(z,\widetilde{\xi})$ is actually a classical symbol of order $-1$; this means by definition that $p(z,\widetilde{\xi})$ admits an asymptotic expansion of the form

$$p(z,\xi) \sim \sum_{j\geq 0} p_{-1-j}(z,\widetilde{\xi}) \tag{3.3.15}$$

for large $\widetilde{\xi}\in\mathbb{R}^{n+1}$, where each $p_m(z,\widetilde{\xi})$ is homogeneous in $\widetilde{\xi}$ of degree $m$. Therefore, $\beta_0^*K \in I^{-1}(M_0^2,\Delta\iota_0)$ (locally identifying a subset of $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ with the bundle $N^*\Delta\iota_0 \simeq {}^0T^*M$).

To show ellipticity we need to show that the principal symbol, given for large $\widetilde{\xi}\in\mathbb{R}^{n+1}$ by $p_{-1}(z,\widetilde{\xi})$ in (3.3.15), is invertible. The full symbol $p$ is given by $\mathcal{F}_Z(K_\varphi)$, but the principal symbol can be computed by taking the Fourier transform of the leading order singularity of $K_\varphi$ at $Z = 0$; the less singular terms contribute to the symbol terms vanishing at least as fast as $|\widetilde{\xi}|_{\overline{g}}^{-2}$ as $|\widetilde{\xi}|_{\overline{g}} \to \infty$, since the symbol is classical. Now one has $\mathcal{F}_Z(|Z|_g^{-n}) = C_n|\widetilde{\xi}|_{\overline{g}}^{-1}$ for $C_n > 0$ in the sense of tempered distributions, hence the principal symbol is $\sigma_0^{-1}(\mathcal{N}_g) = C_n|\widetilde{\xi}|_{\overline{g}}^{-1}$ for large $\widetilde{\xi} \in \mathbb{R}^{n+1}$. Using the identification of $N^*\Delta\iota_0$ with ${}^0T^*M$, and the fact that the latter is trivialized by $\{dz^j/x\}$ near $\partial M$ we can write invariantly $\sigma_0^{-1}(\mathcal{N}_g)(z,\xi) = C_n|\xi|_g^{-1}$, $(z,\xi) \in {}^0T^*M$; this agrees with the principal symbol computed in [SU04]. Note that for $|\xi|_g$ bounded the principal symbol is smooth, since (3.3.13) is compactly supported in $Z$; the singularity at $\xi = 0$ is an artifact of computing the Fourier transform of the non-compactly supported leading order term in (3.3.14). In fact, for bounded $|\xi|_g$ the principal symbol can be freely modified as long as it stays smooth; any such smooth modification will yield the same operator modulo $\Psi_0^{-\infty}(M)$. Due to this fact and since $g$ defines a smooth and non-degenerate quadratic form in the fibers of ${}^0T^*M$, $\sigma_0^{-1}(\mathcal{N}_g)$ is invertible on ${}^0T^*\overline{M}$. We have thus shown that $\mathcal{N}_g \in \Psi_0^{-1,n,n}(M)$ and is elliptic, completing the proof. □

By Propositions 3.3.4 and 3.2.4 it follows immediately that for $s \geq 0$

$$\mathcal{N}_g : x^\delta H_0^s(M;\Omega_0^{1/2}) \to x^{\delta'} H_0^{s+1}(M;\Omega_0^{1/2})$$

is bounded if $\delta > -n/2$, $\delta' < n/2$ and $\delta' \leq \delta$. We can now prove a continuity property for the X-ray transform showing that one can extend it to larger weighted $L^2$ spaces than the ones that appeared in Section 3.1:

**Corollary 3.3.5.** *Let $(\mathring{M}^{n+1}, g)$ be a simple AH manifold. If $\delta' < \delta$, $\delta' < 0$ and $\delta > -n/2$ the X-ray transform is bounded:*

$$I : x^\delta L^2(M; dV_g) \to \langle\eta\rangle_h^{-\delta'} L^2(\partial_- S^*M, d\lambda_\partial).$$

*Proof.* We will show that for $\delta$, $\delta'$ as in the statement there exists a constant $C$ such that for any $f \in C_c^\infty(\mathring{M})$ one has

$$\|If\|_{x^{\delta'}L^2(S\mathring{M};d\lambda)} \leq C\|f\|_{x^\delta L^2(M;dV_g)}. \tag{3.3.16}$$

Since for $f \in C_c^\infty(\mathring{M})$ one has $If \in C^\infty_{\overline{X}}(S^*M) \cap x^{\delta'}L^2(S^*\mathring{M}; d\lambda)$ as explained in the proof of Lemma 3.1.1, the latter applies for $If$ to show that if (3.3.16) is known then one has $\|If\|_{\langle\eta\rangle_h^{-\delta'}L^2(\partial_- S^*M, d\lambda_\partial)} \le C\|f\|_{x^\delta L^2(M; dV_g)}$, yielding the result by density.

First let $0 < \varepsilon < \min\{\delta + n/2, \delta - \delta'\}$ and note that for each fixed $\varepsilon$ the expression $I(x^{2\varepsilon})(z,\xi) = \int_{\mathbb{R}} x^{2\varepsilon} \circ \varphi_t(\xi)dt$ is uniformly bounded on $S^*\mathring{M}$, by the proof of Lemma 3.1.1. Now for $f \in C_c^\infty(\mathring{M})$ apply Cauchy-Schwarz to find

$$\begin{aligned}
\|If\|^2_{x^{\delta'}L^2(S\mathring{M};d\lambda)} &= \int_{S^*\mathring{M}} x^{-2\delta'} |If(z,\xi)|^2 d\lambda \\
&= \int_{S^*\mathring{M}} x^{-2\delta'} \Big| \int_{\mathbb{R}} f(\varphi_t(z,\xi))dt \Big|^2 d\lambda \\
&\le \int_{S^*\mathring{M}} x^{-2\delta'} \int_{\mathbb{R}} x^{2\varepsilon} \circ \varphi_t(z,\xi)dt \int_{\mathbb{R}} |(x^{-\varepsilon}f)(\varphi_t(z,\xi))|^2 dt \; d\lambda \\
&\le C \int_{S^*\mathring{M}} x^{-2\delta'} \int_{\mathbb{R}} |(x^{-\varepsilon}f)(\varphi_t(z,\xi))|^2 dt \; d\lambda \\
&= C \int_{\mathring{M}} x^{-2\delta'} \int_{S_z^*\mathring{M}} I\big(x^{-2\varepsilon}|f|^2\big) d\mu_g \; dV_g(z) \\
&= C \|\mathcal{N}_g(x^{-2\varepsilon}|f|^2)\|_{x^{2\delta'}L^1(M;dV_g)}.
\end{aligned}$$

Now if $\delta'' := \delta - \varepsilon$ the choice of $\varepsilon$, $\delta$ and $\delta'$ imply that $2\delta' < 0$, $2\delta'' > -n$ and $2\delta' \le 2\delta''$. On the other hand, an argument similar to the one of Proposition 3.2.4, though simpler (also see Appendix 1 in [He19]), shows that if $P \in \Psi_0^{-1,\mathcal{E}}(M)$ with $\mathrm{Re}(E_{10}) > n + \sigma'$, $\mathrm{Re}(E_{01}) > -\sigma$ and $\sigma' - \sigma \le \mathrm{Re}(E_{11})$ then $P : x^\sigma L^1(M; dV_g) \to x^{\sigma'}L^1(M; dV_g)$ is bounded. Hence the fact that $\mathcal{N}_g \in \Psi_0^{-1,n,n}(M)$ implies that

$$\|\mathcal{N}_g(x^{-2\varepsilon}|f|^2)\|_{x^{2\delta'}L^1(M;dV_g)} \le C\|x^{-2\varepsilon}|f|^2\|_{x^{2\delta''}L^1(M;dV_g)} = C\|f\|^2_{x^\delta L^2(M;dV_g)}$$

and this finishes the proof. □

**Remark 3.3.6.** *By Corollary 3.3.5 and* (3.1.5) *one also has that* $I^* : \langle\eta\rangle_h^{\delta'}L^2(\partial_- S^*M, d\lambda_\partial) \to x^{-\delta}L^2(M; dV_g)$ *is bounded for* $\delta' < \delta$, $\delta' < 0$ *and* $\delta > -n/2$.

### 3.4 The Model Operator

In this section we show that the model operator of $\mathcal{N}_g$ at a point $p \in \partial M$ can be identified with the normal operator $\mathcal{N}_h$ on the Poincaré hyperbolic ball $(\mathbb{B}^{n+1}, h)$. This operator was studied in [BC91] and explicit inversion formulas was computed for it using the spherical Fourier transform; using those formulas we will show that its inverse lies in $\Psi_0^{1,n+1,n+1}(\overline{\mathbb{B}^{n+1}})$. In what follows we always assume that a choice of coordinates has been made with respect to a point of interest $p \in \partial M$, such that the hyperbolic metric $h_p$ induced by $g$ on $T_p^+M$ (see Section 3.2) takes the form $h_p = u^{-2}(du^2 + |dw|^2)$ with respect to induced linear coordinates $(u, w)$ on $T_p^+M$.

The following is an analog of Proposition 2.17 in [MM87], which shows that for each $p \in \partial M$ the model operator of the Laplacian corresponding to an AH metric $g$ on $\mathring{M}$ is the hyperbolic Laplacian on $(T_p^+M, h_p)$:

**Proposition 3.4.1.** *For any $p \in \partial M$ the model operator $N_p(\mathcal{N}_g)$ on $T_p^+M$ is given by $\mathcal{N}_{h_p}$, the normal operator corresponding to the X-ray transform on $T_p^+M$ endowed with the hyperbolic metric $h_p$.*

*Proof.* As already discussed in detail in Section 3.2, for each $p \in \partial M$ the hyperbolic metric $h_p$ on $T_p^+M$ induces the metrics $h_p^l$ and $h_p^r$ on $\mathring{B}_{11}\big|_p$, which are by construction isometric to each other. We claim that it suffices to show that for $q \in \mathring{B}_{11}\big|_p$, $F_p(\mathcal{N}_g)(q) = \beta_0^* K_{\mathcal{N}_g}\big|_{\mathring{B}_{11}|_p}(q) = K_{\mathcal{N}_{h_p^r}}(e_p, q) = K_{\mathcal{N}_{h_p^l}}(e_p, q)$ for the hyperbolic normal operators $\mathcal{N}_{h_p^l}$, $\mathcal{N}_{h_p^l}$. Indeed, as we will discuss in more detail in the proof of Proposition 3.4.2 below, for the normal operator on hyperbolic space one has that $K_{\mathcal{N}_h}(q, q')$ is a function of the hyperbolic distance function $\rho_h(q, q')$. Recall that we can always arrange that in terms of coordinates $(\widetilde{x}, \widetilde{y}, s, W)$ and $(x, y, t, Y)$ described earlier we have $h_p^l = s^{-2}(ds^2 + |dW|^2)$ and $h_p^r = t^{-2}(dt + |dY|^2)$ respectively. Using the explicit formula for the hyperbolic distance function on the half space model we can see that $\rho_{h_p^l}\big((1,0), (s, W)\big) = \rho_{h_p^l}\big((1,0), (x/\widetilde{x}, (y - \widetilde{y})/\widetilde{x})\big) = \rho_{h_p^l}\big((x, y), (\widetilde{x}, \widetilde{y})\big)$ and analogously for $h_p^r$. Using this fact and computation similar to the one in Lemma 3.2.5

we find with a change of variables that for $f \in C_c^\infty(T_p^+M)$ we have

$$(K_{\mathcal{N}_{h_p^l}}(e_p,\cdot) * f)(v) = \int K_{\mathcal{N}_{h_p}}(v,\widetilde{v}) f(\widetilde{v}) dV_{h_p}(\widetilde{v}),$$

which implies the statement of the proposition.

It is more convenient given the setup of Lemma 3.3.1 to show that $F_p(\mathcal{N}_g)(q) = K_{\mathcal{N}_{h_p^r}}(e_p,q)$. The fact that $K_{\mathcal{N}_{h_p^r}}(e_p,q) = K_{\mathcal{N}_{h_p^l}}(e_p,q)$ will then follow by noting that $\rho_{h_p^l}(e_p,q) = \rho_{h_p^r}(e_p,q)$ and $K_{\mathcal{N}_h}$ only depends on $\rho_h$ for a hyperbolic metric $h$, as already mentioned. We use copies $z = (x,y)$, $\widetilde{z} = (\widetilde{x},\widetilde{y})$ of the same coordinate system near $p \in \partial M$ such that in terms of coordinates $\overline{Z} = (t,Y)$ on $\mathring{B}_{11}\big|_p$, $h_p^r = t^{-2}(dt^2 + |dY|^2)$. By the proof of Lemma 3.3.3, we can rewrite the function $K_{\mathcal{N}_g}$ in (3.3.6) as

$$K_{\mathcal{N}_g}(z,\widetilde{z}) = \frac{2\sqrt{\det g(z)}\,\big|\det \partial_{\widetilde{z}}(\exp_z^{-1}(\widetilde{z}))\big|}{\rho^n(z,\widetilde{z})\sqrt{\det g(\widetilde{z})}}, \tag{3.4.1}$$

where the matrix $\partial_{\widetilde{z}}(\exp_z^{-1}(\widetilde{z}))$ is computed with $\exp_z^{-1}(\widetilde{z})$ written in terms of coordinates on $T_zM$ determined by the vectors $\partial_{z^0},\ldots,\partial_{z^n}$. Recall the map $\Phi(z,\xi) = \left(z, \exp_z\left(\xi^\#\right)\right) = (z, \pi\circ \varphi_1(z,\xi))$ and its extension $\widetilde{\Phi}(z,\xi) = \pi^\Phi \circ \widetilde{\varphi}_1 \circ \psi(z,\xi)$ from Lemma 3.3.1. If $u\partial_x + w^\tau \partial_{y^\tau} \in TM^\circ$ denotes a generic vector (so $v = (u, w^\alpha)$ are induced fiber coordinates) we can see that (3.3.3) implies that on $\mathring{M}^2$

$$d_{(z,\xi)}\widetilde{\Phi} = \begin{pmatrix} Id & 0 \\ * & d_v\left(\widetilde{z}\circ \exp_z(v)\right)\big|_{v=\xi^\#}\overline{g}^{-1}(z)\end{pmatrix}, \quad (z,\xi) \in ({}^0T^*M)^\circ.$$

Thus on $(M_0^2)^\circ$

$$\begin{aligned}\beta_0^* \det(d_{\widetilde{z}} \exp_z^{-1}(\widetilde{z})) &= \beta_0^* \det(d_v\, \widetilde{z}\circ \exp_z\big|_{v=(\Phi^{-1}(z,\widetilde{z}))^\#})^{-1} = \det\left(d_v\, \widetilde{z}\circ \exp_z\big|_{v=(\widetilde{\Phi}^{-1})^\#}\right)^{-1} \\ &= (\det \overline{g}(z))^{-1}\left(\det d_{(z,\xi)}\widetilde{\Phi}\right)^{-1} \circ \widetilde{\Phi}^{-1} = (\det \overline{g}(z))^{-1}\left(\det d_\xi(\overline{Z}\circ\widetilde{\Phi})\right)^{-1} \circ \widetilde{\Phi}^{-1}.\end{aligned}$$

Now ${}^0T^*M$ can be identified with $N^*\Delta\iota_0$ and the fiber of the latter over $e_p$ with $T^*_{e_p}\mathring{B}_{11}\big|_p$, hence ${}^0T_p^*M \simeq T^*_{e_p}(\mathring{B}_{11}\big|_p)$. Using this identification, $\widetilde{\Phi}\big|_{{}^0T_p^*M} = \Phi_{h_p^r} : T^*_{e_p}\mathring{B}_{11}\big|_p \to \mathring{B}_{11}\big|_p$, $(e_p,\xi) \mapsto \pi\circ\widetilde{\varphi}_1^{h_p^r}(e_p,\xi)$, where $\widetilde{\varphi}_t^{h_p^r}$ is the flow of the Hamiltonian $X^{h_p^r}$ generated on $T^*(\mathring{B}_{11}\big|_p)$ by the Lagrangian $\mathcal{L}_{h_p^r} = t^2(\tau^2 + \Upsilon^2)$. This follows from the proof of Proposition 23 in [CH16]:

the lift of the right Hamiltonian $X_R$ to ${}^{\Phi}T^*M_0^2$ (which generates the flow $\widetilde{\varphi}_t$) is tangent to $\mathring{B}_{11}\big|_p$ and, on its flowout from $N^*\Delta\iota_0 \cap \{\mathcal{L}_g = \text{const.}\}$, its restriction to $\mathring{B}_{11}\big|_p$ can be identified with $X^{h_p^r}$ viewed as a vector field on ${}^0T^*B_{11}\big|_p$.

Restricting to $\mathring{B}_{11}\big|_p$ we obtain, using that $\det(\overline{g}(p)) = 1$ with our normalization,

$$\beta_0^* \det(d_{\widetilde{z}} \exp_z^{-1}(\widetilde{z}))\big|_{B_{11}\big|_p} = \left(\det d_\xi \left(\overline{Z} \circ \Phi_{h_p^r}\right)\right)^{-1} \circ \Phi_{h_p^r}^{-1} = \det\left(d_{\overline{Z}} \widetilde{\exp}_{e_p}^{-1}(\overline{Z})\right), \quad (3.4.2)$$

where in (3.4.2) $\xi$ denote fiber variables on $T_{e_p}^*\mathring{B}_{11}\big|_p$ and $\widetilde{\exp}$ is the $h_p^r$-exponential map. Now by Proposition 24 in [CH16] $\beta_0^*\rho$ restricts to the front face as the $h_p^r$-distance between $(1,0)$ and $\overline{Z} = (t, Y)$. Moreover,

$$\beta_0^* \frac{\sqrt{\det g(z)}}{\sqrt{\det g(\widetilde{z})}} = \frac{x^{-n-1}\sqrt{\det \overline{g}(x,y)}}{(tx)^{-n-1}\sqrt{\det \overline{g}(tx, y + xY)}} \Rightarrow \beta_0^* \frac{\sqrt{\det g(z)}}{\sqrt{\det g(\widetilde{z})}} = \frac{1}{t^{-n-1}} \text{ at } p.$$

Combining the restrictions of the various factors of (3.4.1) completes the proof. $\square$

As mentioned in Section 3.2, for each $p \in \partial M$ the model operator $\mathcal{N}_{h_p}$ can be equivalently realized as an operator acting on $C_c^\infty(\mathbb{B}^{n+1})$, where $(\mathbb{B}^{n+1}, h)$ is the Poincaré ball with metric $h = \frac{4|dz|^2}{(1-|z|^2)^2}$), upon making a choice of coordinates that identifies $(T_p^+M, h_p)$ with $(\mathbb{H}^{n+1}, h)$ and conjugating by the Cayley transform. For the next proof we write $\mathbb{B}$ instead of $\mathbb{B}^{n+1}$ (i.e. without a superscript for the dimension). The following proposition is essentially an immediate consequence of the results in [BC91]:

**Proposition 3.4.2.** *For any $p \in \partial M$ the model operator $N_p(\mathcal{N}_g)$ can be identified with the operator $\mathcal{N}_h : C_c^\infty(\mathbb{B}; \Omega_0^{1/2}) \to C^{-\infty}(\mathbb{B}; \Omega_0^{1/2})$ on $(\mathbb{B}^{n+1}, h)$, which for $\delta \in (-n/2, n/2)$ extends continuously to an operator $\mathcal{N}_h : x^\delta L^2(\overline{\mathbb{B}}; \Omega_0^{1/2}) \to x^\delta H_0^1(\overline{\mathbb{B}}; \Omega_0^{1/2})$. The operator $\mathcal{N}_h$ has a two sided inverse $\mathcal{N}_h^{-1} \in \Psi_0^{1,n+1,n+1}(\overline{\mathbb{B}})$ such that $\mathcal{N}_h^{-1}\mathcal{N}_h = \mathcal{N}_h\mathcal{N}_h^{-1} = Id$ on $x^\delta L^2(\overline{\mathbb{B}}; \Omega_0^{1/2})$ for $\delta \in (-n/2, n/2)$.*

*Proof.* For each $p \in \partial M$, $N_p(\mathcal{N}_g) = \mathcal{N}_{h_p}$ on $(T_p^+M, h_p)$ by Proposition 3.4.1, and $\mathcal{N}_{h_p}$ can be identified with $\mathcal{N}_h$ on $(\overline{\mathbb{B}}, h)$ as explained before. Thus by Proposition 3.3.4 $\mathcal{N}_h \in \Psi_0^{-1,n,n}(\overline{\mathbb{B}})$ and the extension statement follows from Proposition 3.2.4. It was observed in [BC91] that

$\mathcal{N}_h$ can be expressed as

$$\mathcal{N}_h f(z) = \int_{\mathbb{B}} \mathcal{R}\big(\rho_h(z,\widetilde{z})\big) f(\widetilde{z}) dV_h(\widetilde{z}), \quad f \in C_c^\infty(\mathbb{B}), \tag{3.4.3}$$

where $\rho_h$ is the geodesic distance function with respect to the hyperbolic metric and

$$\mathcal{R}(r) = \pi^{-n/2} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})} \sinh^{-n}(r).$$

Note that $\mathbb{B}$ is a homogeneous space on which $G = O^+(1, n+1)$ acts by isometries, thus it can be identified with the quotient $G/H_o$, where $H_o \cong O(n+1)$ is the isotropy group of the origin $o \in \mathbb{B}$. Hence (3.4.3) can be interpreted as convolution by a locally integrable radial ($H_o$-invariant) function: for $f \in C_c^\infty(\mathbb{B})$

$$\mathcal{N}_h f(gH_o) = \mathcal{R} * f(gH_o) = \int_{\mathbb{B}} \mathcal{R}(\widetilde{g}^{-1} g H_o) f(\widetilde{g} H_o) dV_h(\widetilde{g} H_o), \quad z = gH_o,\ \widetilde{z} = \widetilde{g} H_o,$$

where above and in what follows by abuse of notation we identify radial functions and distributions on $\mathbb{B}$ with ones on $[0,\infty)$, writing for instance $\mathcal{R}(z) = \mathcal{R}(\rho_h(z,o))$ for $z \in \mathbb{B}$.

An exact left inverse for $\mathcal{N}_h$ is computed in [BC91] (Theorems 4.2, 4.3, 4.4): if $\Delta$ denotes the hyperbolic Laplacian with principal symbol $-|\xi|_h^2$, one has

$$C_n p(\Delta) S_n \,\mathcal{N}_h = Id \quad \text{ on } C_c^\infty(\mathbb{B}).$$

Here $C_n$ is an explicit constant, $p(t) = -(t+n-1)$ and $S_n$ is given by convolution by the locally integrable radial kernel

$$\mathcal{S}_n(r) = \begin{cases} \coth(r) - 1, & n = 1 \\ \sinh^{-n}(r)\cosh(r), & n \geq 2 \end{cases}, \tag{3.4.4}$$

that is,

$$S_n f(z) = \mathcal{S}_n * f(z) = \int_{\mathbb{B}} \mathcal{S}_n\big(\rho_h(z,\widetilde{z})\big) f(\widetilde{z}) dV_h(\widetilde{z}), \quad f \in C_c^\infty(\mathbb{B}).$$

The fact that $C_n p(\Delta) S_n$ is also a right inverse for $\mathcal{N}_h$ follows by tracing through the proofs of Theorems 4.2-4.5 in [BC91]. They use the spherical Fourier transform of a radial distribution, given by $\widehat{f}(\lambda) = \int_{\mathbb{B}} f(\widetilde{z})\phi_{-\lambda}(\widetilde{z})dV_h(\widetilde{z})$ for $\lambda \in \mathbb{R}$, where $\phi_\lambda$ is the radial eigenfunction of $\Delta$ with eigenvalue $-n^2/4 - \lambda^2$ that satisfies $\phi_\lambda(o) = 1$. The spherical Fourier transform is well defined pointwise whenever $f(\widetilde{z})\phi_{-\lambda}(\widetilde{z})$ is integrable; the reason why the formula corresponding to $n = 1$ in (3.4.4) differs from the one corresponding to $n \geq 2$ is exactly to ensure that $\widehat{\mathcal{S}_1}$ is well defined. Their strategy is to show that $(C_n p(\Delta)\mathcal{S}_n * \mathcal{R})\widehat{\ } = \widehat{\delta}$, where $\delta$ is the delta distribution at the origin. Thus the claim reduces to showing that $(\mathcal{R} * C_n p(\Delta)\mathcal{S}_n)\widehat{\ } = \widehat{\delta}$. This in turn follows from the fact that for radial distributions $\mathcal{U}$, $\mathcal{V}$ one has $\widehat{\mathcal{U} * \mathcal{V}}(\lambda) = \widehat{\mathcal{U}}(\lambda)\widehat{\mathcal{V}}(\lambda)$, and also $\widehat{p(\Delta)\mathcal{U}}(\lambda) = -p(-n^2/4 - \lambda^2)\widehat{\mathcal{U}}(\lambda)$, provided the expressions make sense.

Now let $\varphi(x) \in C_c^\infty([0,\infty))$ be identically 1 on $[0,1]$ and identically 0 on $[0,2]^c$ and let

$$\begin{aligned} S_{n;1}f(z) &= \int_{\mathbb{B}} \varphi(\rho_h(z,\widetilde{z}))\mathcal{S}_n\big(\rho_h(z,\widetilde{z})\big)f(\widetilde{z})dV_h(\widetilde{z}) \\ \text{and } S_{n;2}f(x) &= \int_{\mathbb{B}} \big(1-\varphi(\rho_h(x,\widetilde{z}))\big)\mathcal{S}_n\big(\rho_h(z,\widetilde{z})\big)f(\widetilde{z})dV_h(\widetilde{z}) \text{ for } f \in C_c^\infty(\mathbb{B}^n), \end{aligned}$$

so that $S_n = S_{n;1} + S_{n;2}$. Using Proposition 3.3.2 one sees that the Schwartz kernel of $S_{n;1}$ vanishes identically near the left and right faces of the 0-stretched product $\overline{\mathbb{B}}_0^2$; thus together with the last part of the proof of Proposition 3.3.4 analyzing the conormal singularity of $\mathcal{N}_g$ we find that $S_{n;1} \in \Psi_0^{-1}(\overline{\mathbb{B}})$ and hence $p(\Delta)S_{n;1} \in \Psi_0^1(\overline{\mathbb{B}})$ since $p(\Delta) \in \mathrm{Diff}_0^2(\overline{\mathbb{B}})$.

The Laplacian acting on radial distributions is given in terms of geodesic polar coordinates by $\Delta = \partial_r^2 + n\coth(r)\partial_r$ and one checks that for $n \geq 1$

$$p(\Delta)\mathcal{S}_n(r) = -(\Delta + n - 1)\sinh^{-n}(r)\cosh(r) = -n\sinh^{-n-2}(r)\cosh(r). \tag{3.4.5}$$

Since for $f \in C_c^\infty(\mathbb{B})$

$$p(\Delta)S_{n;2}f(z) = \int_{\mathbb{B}} p(\Delta)\big((1-\varphi(r))\mathcal{S}_n(r)\big)\big|_{r=\rho_h(z,\widetilde{z})}f(\widetilde{z})dV_h(\widetilde{z}),$$

(3.4.5) and Proposition 3.3.2 imply that $P(\Delta)S_{n;2} \in \Psi_0^{-\infty,n+1,n+1}(\overline{\mathbb{B}})$. We conclude that $p(\Delta)S_n \in \Psi_0^{1,n+1,n+1}(\overline{\mathbb{B}})$ for all $n \geq 1$ and the spaces on which the inversion is valid follow again from Proposition 3.2.4 by density. $\square$

### 3.5 Parametrix construction and Stability Estimates

**Proposition 3.5.1.** *Let* $(\mathring{M}^{n+1}, g)$ *be a simple AH manifold. There exists an operator* $B$ *such that for* $\delta \in (-n/2, n/2)$ *and* $s \geq 0$

$$B : x^\delta H_0^{s+1}(M; \Omega_0^{1/2}) \to x^\delta H_0^s(M; \Omega_0^{1/2}) \tag{3.5.1}$$

*is bounded and on* $x^\delta H_0^s(M; \Omega_0^{1/2})$ *one has*

$$B\mathcal{N}_g = Id - K, \quad K \in \Psi_0^{-\infty,\mathcal{F}}(M), \quad F_{11} \geq 1, \quad F_{10}, F_{01} \geq n. \tag{3.5.2}$$

*In particular,* $K : x^\delta H_0^s(M; \Omega_0^{1/2}) \to x^\delta H_0^s(M; \Omega_0^{1/2})$ *is compact for such* $\delta$ *and* $s$.

*Proof.* We write $\mathcal{N}_g = A_1 + A_2$, where $A_1 \in \Psi_0^{-1}(M)$, $A_2 \in \Psi_0^{-\infty,n,n}(M)$. By the ellipticity of $\mathcal{N}_g$ (and hence of $A_1$), Theorem 3.8 in [Maz91] shows the existence of $B_1 \in \Psi_0^1(M)$ such that

$$B_1 A_1 = Id - K_1, \quad K_1 \in \Psi_0^{-\infty}(M).$$

Note that $K_1$ is not compact on any weighted Sobolev space $x^\sigma H_0^s(M; \Omega_0^{1/2})$ since its kernel does not vanish at $B_{11}$. Using Proposition 3.2.1 we reach

$$B_1\mathcal{N}_g = Id - K_2, \quad K_2 := K_1 - B_1 A_2 \in \Psi_0^{-\infty,n,n}(M).$$

We now improve the error term to to ensure that its kernel vanishes at the front face. For each $p \in \partial M$, $F_p(K_2) \in \mathcal{A}_{phg}^{n,n}(B_{11}\big|_p)$ and thus $N_p(K_2) \in \Psi_0^{-\infty,n,n}(\overline{\mathbb{B}^{n+1}})$, under the identification of $(T_p^+ M, h_p)$ with $(\mathbb{B}^{n+1}, h)$ using coordinates and the Cayley transform as described before; this identification depends smoothly on $p$. Propositions 3.4.2 and 3.2.1 imply that $N_p(K_2)\mathcal{N}_h^{-1} = N_p(K_2)N_p(\mathcal{N}_g)^{-1} \in \Psi_0^{-\infty,\mathcal{E}}(\overline{\mathbb{B}^{n+1}})$, $E_{10}$, $E_{01} \geq n$, $\mathrm{Re}(E_{11}) \geq 0$. In fact, we can obtain an improvement of the expansions: to see this, use Propositions 3.2.7 and 3.4.2 to write

$$N_p(K_2)N_p(\mathcal{N}_g)^{-1} = N_p(Id - B_1\mathcal{N}_g)N_p(\mathcal{N}_g)^{-1} = N_p(\mathcal{N}_g)^{-1} - N_p(B_1) \in \Psi_0^{1,n+1,n+1}(\overline{\mathbb{B}^{n+1}}).$$

Thus $N_p(K_2)N_p(\mathcal{N}_g)^{-1} \in \Psi_0^{-\infty,\mathcal{E}}(\overline{\mathbb{B}^{n+1}}) \cap \Psi_0^{1,n+1,n+1}(\overline{\mathbb{B}^{n+1}}) \subset \Psi_0^{-\infty,n+1,n+1}(\overline{\mathbb{B}^{n+1}})$. Again using the identification $(T_p^+ M, h_p) \leftrightarrow (\mathbb{B}^{n+1}, h)$, the convolution kernel of $N_p(K_2)N_p(\mathcal{N}_g)^{-1}$ is a polyhomogeneous (and in fact smooth) function in $\mathcal{A}_{phg}^{n+1,n+1}(B_{11}\big|_p)$. According to (3.2.17), it can be extended off of $B_{11}$ smoothly to produce an operator $B_2 \in \Psi_0^{-\infty,n+1,n+1}(M)$ such that at each $p \in \partial M$, $F_p(B_2)$ agrees with the convolution kernel of $N_p(K_2)N_p(\mathcal{N}_g)^{-1}$. By Lemma 3.2.5 and Proposition 3.2.7 this implies that $F_p(B_2\mathcal{N}_g) = F_p(K_2)$. Setting $B = B_1 + B_2 \in \Psi_0^{1,n+1,n+1}(M)$ and using Proposition 3.2.1 we find

$$B\mathcal{N}_g = Id - K, \quad K \in \Psi_0^{-\infty,\mathcal{F}}(M),$$
$$F_{11} = \overline{\{(1,0)\}} \cup \overline{\{(2n+1,1)\}}, \quad F_{10} = F_{01} = \overline{\{(n,0)\}} \cup \overline{\{(n+1,1)\}}.$$

As already stated earlier, by Proposition 3.2.4 one has that for $s \geq 0$, $\mathcal{N}_g : x^\delta H_0^s(M;\Omega_0^{1/2}) \to x^{\delta'} H_0^{s+1}(M;\Omega_0^{1/2})$ is bounded provided $\delta > -n/2$, $\delta' < n/2$ and $\delta' \leq \delta$. Moreover, $B : x^{\delta'} H_0^{s+1}(M;\Omega_0^{1/2}) \to x^{\delta''} H_0^s(M;\Omega_0^{1/2})$ is bounded provided $\delta' > -n/2-1$, $\delta'' < n/2+1$ and $\delta'' \leq \delta'$. Hence choosing $\delta = \delta' = \delta'' \in (-n/2, n/2)$ we obtain (3.5.1) and (3.5.2). Moreover, for such choice of $\delta$ one can choose $\widetilde{\delta}$ such that $\delta < \widetilde{\delta} < \min\{n/2, \delta+1\}$, and $\widetilde{s} > s$ to guarantee that

$$K : x^\delta H_0^s(M;\Omega_0^{1/2}) \to x^{\widetilde{\delta}} H_0^{\widetilde{s}}(M;\Omega_0^{1/2})$$

is bounded, implying that $K : x^\delta H_0^s(M;\Omega_0^{1/2}) \to x^\delta H_0^s(M;\Omega_0^{1/2})$ is compact, as claimed. $\square$

Proposition 3.5.1 together with Proposition 3.2.4 imply that for $\delta \in (-n/2, n/2)$

$$x^\delta L^2(M;\Omega_0^{1/2}) \cap \ker \mathcal{N}_g \subset \bigcap_{m\in\mathbb{R}} x^\delta H_0^m(M;\Omega_0^{1/2}) =: x^\delta H_0^\infty(M;\Omega_0^{1/2}) \subset C^\infty(\mathring{M};\Omega_0^{1/2}). \quad (3.5.3)$$

We will now show using a technique shown to us by Rafe Mazzeo that the functions in (3.5.3) also have polyhomogeneous expansions at the boundary. We start by showing tangential regularity (here we work with functions as opposed to half densities for convenience).

**Lemma 3.5.2.** *Let $u \in x^\delta L^2(M;dV_g) \cap \ker \mathcal{N}_g$, with $\delta \in (-n/2, n/2)$. Then*

$$u \in x^\delta H_b^\infty(M;dV_g) := \{u \in x^\delta L^2(M;dV_g) :$$
$$V_1 \cdots V_m u \in x^\delta L^2(M;dV_g), \text{ for all } m \geq 0,\ V_j \in \mathcal{V}_b(M)\}.$$

*Proof.* Smoothness of $u$ in $\mathring{M}$ was already remarked in (3.5.3). By analogy with $\mathrm{Diff}_0^m(M)$, below we write $\mathrm{Diff}_b^m(M)$ for the differential operators consisting of finite sums of at most $m$-fold products of vector fields in $\mathcal{V}_b(M)$. Let $u$ be as in the statement and $\delta \in (-n/2, n/2)$. We will show that for any $m \geq 0$, if $Pu \in x^\delta L^2(M; dV_g)$ whenever $P \in \mathrm{Diff}_b^m(M)$, then $P'u \in x^\delta L^2(M; dV_g)$ whenever $P' \in \mathrm{Diff}_b^{m+1}(M)$. Since $u \in x^\delta L^2(M; dV_g)$ by assumption, this suffices to prove the lemma.

The claim will be shown by induction. To motivate the inductive hypothesis, first observe that a vector field $V \in \mathcal{V}_b(M)$ lifts from the left factor of $M^2$ to a vector field on $M_0^2$ tangent to the interior of the side faces, which is either smooth everywhere on $M_0^2$ or it blows up at the front face with order 1. Thus by Proposition 3.5.1 we have $Vu = VKu$ with $VK \in \Psi_0^{-\infty, F_{10}, F_{01}, F'_{11}}(M)$, $F'_{11} = F_{11} - 1 \geq 0$. This implies that $Vu \in x^\delta L^2(M; dV_g)$, by Proposition 3.2.4.

Now fix $m \geq 0$ and assume that if $P \in \mathrm{Diff}_b^{m+1}(M)$, then $Pu$ can be written as a finite sum

$$Pu = \sum_j Q_j^{(m)} P_j^{(m)} u, \quad Q_j^{(m)} \in \Psi_0^{-\infty, F_{10}, F_{01}, F'_{11}} \text{ and } P_j^{(m)} \in \mathrm{Diff}_b^m(M). \tag{3.5.4}$$

As we already showed, the claim is true for $m = 0$. We will show that (3.5.4) holds for $m+1$. Any operator in $\mathrm{Diff}_b^{m+2}(M)$ can be written as a finite sum of the form $\sum_j V_j P_j$, where $V_j \in \mathcal{V}_b(M)$ and $P_j \in \mathrm{Diff}_b^{m+1}(M)$. Thus it suffices to differentiate (3.5.4) by $V \in \mathcal{V}_b(M)$ and show that it has the required form. We find

$$VPu = \sum_j V Q_j^{(m)} P_j^{(m)} u = \sum_j \big( Q_j^{(m)} V P_j^{(m)} u - [Q_j^{(m)}, V] P_j^{(m)} u \big).$$

By Proposition 3.30 in [Maz91], $[Q, V] \in \Psi_0^{-\infty, \mathcal{E}}(M)$ for $Q \in \Psi_0^{-\infty, \mathcal{E}}(M)$ and $V \in \mathcal{V}_b(M)$. Thus $[Q_j^{(m)}, V] \in \Psi_0^{-\infty, F_{10}, F_{01}, F'_{11}}(M)$ for all $j$. Since $VP_j^{(m)}$, $P_j^{(m)} \in \mathrm{Diff}_b^{m+1}(M)$ we obtain (3.5.4) for $m+1$.

Now the fact that for any $m \geq 0$, $Pu \in x^\delta L^2(M; dV_g)$ for $P \in \mathrm{Diff}_b^m(M)$ implies that $P'u \in x^\delta L^2(M; dV_g)$ for $P' \in \mathrm{Diff}_b^{m+1}(M)$ follows immediately by (3.5.4) and Proposition 3.2.4. $\square$

We will use the Mellin transform to show the existence of polyhomogeneous expansion at the boundary for elements in the nullspace of $\mathcal{N}_g$. We briefly recall its definition and main properties. Below we write $\mathbb{R}^+ = (0, \infty)$.

**Definition 3.5.3.** *If $f \in C_c^\infty(\mathbb{R}^+)$ and $\zeta \in \mathbb{C}$ we define the Mellin Transform of $f$ by*

$$f_{\mathcal{M}}(\zeta) = \int_0^\infty x^\zeta f(x) \frac{dx}{x}.$$

By the fact that $f_{\mathcal{M}}(\zeta) = \mathcal{F}(f(\exp(\,\cdot\,)))(i\zeta)$ for $\zeta$ imaginary, we see that for $f \in C_c^\infty(\mathbb{R}^+)$ the Mellin transform is rapidly decaying along each line $\zeta = \alpha + i\eta$, as $\mathbb{R} \ni \eta \to \pm\infty$ where $\alpha \in \mathbb{R}$ is constant. Moreover, there is automatically a Plancherel type theorem: that is, we obtain an isomorphism

$$\mathcal{M} : L^2(\mathbb{R}^+; \frac{dx}{x}) \to L^2(\{\mathrm{Re}(\zeta) = 0\}; |d\zeta|).$$

More generally, the Mellin transform induces an isomorphism

$$\mathcal{M} : x^\delta L^2\big(\mathbb{R}^+; \frac{dx}{x}\big) \to L^2(\{\mathrm{Re}(\zeta) = -\mathrm{Re}(\delta)\}; |d\zeta|)$$

with inverse given by

$$u(x) = \frac{1}{2\pi} \int_{\mathrm{Re}(\zeta) = -\mathrm{Re}(\delta)} x^{-\zeta} u_{\mathcal{M}}(\zeta) |d\zeta|.$$

Moreover, by the Paley-Wiener theorem if $u \in x^\delta L^2\big(\mathbb{R}^+; \frac{dx}{x}\big)$ and $\operatorname{supp} u \subset [0, 1)$ then $u_{\mathcal{M}}$ extends to a holomorphic function on the half plane $\{\mathrm{Re}(\zeta) > -\mathrm{Re}(\delta)\}$, uniformly in $L^2(\{\mathrm{Re}(\zeta) = \alpha\}; |d\zeta|)$ for $\alpha \geq -\mathrm{Re}(\delta)$, that is, $\sup_{\alpha \geq -\mathrm{Re}(\delta)} \|u_{\mathcal{M}}\|_{L^2(\{\mathrm{Re}(\zeta)=\alpha\}, |d\zeta|)} \leq C$. On the other hand, if $u \in L^2(\mathbb{R}^+; \frac{dx}{x})$ with $\operatorname{supp} u \subset\subset (0, \infty)$ then $u_{\mathcal{M}}(\zeta)$ extends to be entire, with $|u_{\mathcal{M}}(\zeta)| \leq A e^{B|\mathrm{Re}(\zeta)|}$ for constants $A, B$ depending on $u$. By analogy with the Fourier transform, we also have $(x\partial_x u)_{\mathcal{M}}(\zeta) = -\zeta u_{\mathcal{M}}(\zeta)$ on the half plane $\{\mathrm{Re}(\zeta) \geq -\mathrm{Re}(\delta)\}$ provided $u \in x^\delta H_b^1(\mathbb{R}^+; \frac{dx}{x}) := \{u \in x^\delta L^2(\mathbb{R}^+; \frac{dx}{x}) : x\partial_x u \in x^\delta L^2(\mathbb{R}^+; \frac{dx}{x})\}$ with $\operatorname{supp} u \subset [0, 1)$. Moreover, if $\varphi \in C_c^\infty\big([0, \infty)\big)$ is identically 1 near 0 then $(x^\delta |\log(x)|^k \varphi)_{\mathcal{M}}(\zeta)$ is holomorphic on the half plane $\{\mathrm{Re}(\zeta) > -\mathrm{Re}(\delta)\}$ for $k$ non-negative integer, and using an integration by parts one sees that it extends meromorphically on $\mathbb{C}$, with a pole of order $k + 1$ at $\zeta = -\delta$.

If $M$ is a compact manifold with boundary one can use a product decomposition $[0,\varepsilon)_x \times \partial M$ of a collar neighborhood of $\partial M$ and compute the Mellin transform in the $x$ variable for polyhomogeneous conormal functions supported near $\partial M$. If $\varphi \in C^\infty(M)$ is supported near $\partial M$ and $u \in \mathcal{A}^E_{phg}(M)$, then $(u\varphi(x))_{\mathcal{M}}$ is meromorphic on $\mathbb{C}$ with poles of order $p+1$ at $\zeta = -s-\ell$ and values in $C^\infty(\partial M)$ for each $(s,p) \in E$ and for $\ell \in \mathbb{N}_0 = \{0,1,\dots\}$. The fact that the space $\mathcal{A}^{\mathcal{E}}_{phg}(M)$ is invariantly defined, as already remarked earlier, implies that the analyticity properties of $(\varphi u)_{\mathcal{M}}$ are invariantly defined.

Before we show the existence of an asymptotic expansion for elements in the nullspace of $\mathcal{N}_g$ we show a lemma about index sets.

**Lemma 3.5.4.** *Let $E_1$, $E_2$, $F \subset \mathbb{C}\times\mathbb{N}_0$ be index sets satisfying* (3.2.2). *Then $(E_1 \overline{\cup} E_2) + F \subset (E_1 + F)\overline{\cup}(E_2 + F)$.*

*Proof.* First note that $(E_1 \cup E_2) + F = (E_1 + F) \cup (E_2 + F) \subset (E_1 + F)\overline{\cup}(E_2 + F)$. Now suppose that $(s, p_1 + p_2 + 1) \in E_1 \overline{\cup} E_2$, where $(s, p_1) \in E_1$ and $(s, p_2) \in E_2$ and let $(\widetilde{s}, \widetilde{p}) \in F$. Then $(s + \widetilde{s}, (p_1 + \widetilde{p}) + (p_2 + \widetilde{p}) + 1) \in (E_1 + F)\overline{\cup}(E_2 + F)$, so it is also the case that $(s, p_1 + p_2 + 1) + (\widetilde{s}, \widetilde{p}) = (s + \widetilde{s}, p_1 + p_2 + \widetilde{p} + 1) \in (E_1 + F)\overline{\cup}(E_2 + F)$ by (3.2.2) and we have shown the claim. $\square$

**Remark 3.5.5.** *In general one does not have $(E_1 \overline{\cup} E_2) + F = (E_1 + F)\overline{\cup}(E_2 + F)$. For instance consider the index sets $E_1 = \overline{\{(1,10)\}}$, $E_2 = \overline{\{(1/2,0)\}}$ and $F = \overline{\{(1/2,5),(0,0)\}}$. Then $(1,16) \in (E_1 + F)\overline{\cup}(E_2 + F) \setminus \big((E_1 \overline{\cup} E_2) + F\big)$.*

**Proposition 3.5.6.** *Let $u \in x^\delta L^2(M; dV_g) \cap \ker \mathcal{N}_g$, with $\delta \in (-n/2, n/2)$. Then $u \in \mathcal{A}^E_{phg}(M)$ with $E = \overline{\bigcup}_{j\geq 0}(F_{10} + jF_{11})$, where $F_{10}$, $F_{11}$ are the index sets in* (3.5.2) *and $jF_{11} = \sum_{i=1}^{j} F_{11}$. Note that $F_{10} + jF_{11} \geq n + j$ and hence $E$ is an index set.*

*Proof.* By (3.5.3), any $u$ as in the statement is smooth in $\mathring{M}$, hence it suffices to show the existence of an asymptotic expansion at the boundary for $u$. We first show that if $u \in x^\delta L^2(M; dV_g) \cap \ker \mathcal{N}_g$ for some $\delta \in (-n/2, n/2)$ then $u \in x^{\delta'} H^\infty_0(M; dV_g)$ for all $\delta' < n/2$. Since $u = Ku$, the mapping properties of $K$ (by (3.5.2) and Proposition 3.2.4) imply that

$u \in x^{\delta_1} H_0^\infty(M; dV_g)$ provided $\delta_1 < n/2$, $\delta_1 \leq \delta + 1$, that is, the existence of a parametrix allows us to obtain an improvement in the decay of $u$. Using the improved decay and $u = Ku$ $j$ times, we inductively find that in fact $u \in x^{\delta_j} H_0^\infty(M; dV_g)$, provided $\delta_j < n/2$ and $\delta_j \leq \delta + j$, that is, taking $j$ sufficiently large we conclude that $u \in x^{\delta'} H_0^\infty(M; dV_g)$ for $\delta' < n/2$. Equivalently, $u \in x^\tau H_0^\infty\big(M; \frac{|dxdy|}{x}\big)$, where $\tau < n$. (For the remaining part of the argument we prefer to use the measure induced by $\Omega_b(M)$, which by abuse of notation write as $\frac{|dxdy|}{x}$, due to its more natural behavior with respect to the Mellin transform.) By Lemma 3.5.2, $u \in x^\tau H_b^\infty\big(M; \frac{|dxdy|}{x}\big)$ for $\tau < n$.

Functions in $x^\tau H_b^\infty(M; \frac{|dxdy|}{x})$ supported near $\partial M$ can be identified with functions in $\bigcap_{k,\ell \in \mathbb{N}_0} x^\tau H_b^k(dx/x; H^\ell(\partial M))$, where $x^\tau H_b^k(dx/x; H^\ell(\partial M))$ is the space of $v : \mathbb{R}^+ \to H^\ell(\partial M)$ that are almost everywhere on $\mathbb{R}^+$ $k$ times Fréchet differentiable (and supported near 0), and $\|x^{-\tau}(x\partial_x)^j v\|_{H^\ell(\partial M)} \in L^2(dx/x)$ for $j = 0, \ldots, k$. Therefore, if $\varphi \in C_c^\infty(M)$ is supported in a sufficiently small neighborhood of $\partial M$ and identically 1 near $\partial M$, by taking the Mellin transform in $x$ we find that $(\varphi u)_{\mathcal{M}}(\zeta)$ is holomorphic in the half plane $\{\mathrm{Re}(\zeta) > -\tau\}$, with values in smooth functions with respect to $y$ and with the $L^2(\{\mathrm{Re}(\zeta) = \alpha\}; |d\zeta|)$ norm of $\|(\varphi u)_{\mathcal{M}}\|_{H^\ell(\partial M)}$ being uniformly bounded for $\alpha \geq -\tau$ for each $\ell$.

We now recover the leading order term in the expansion of $u$ at $\partial M$. We first make the observation that localizing $K$ near the boundary from the left does not alter its index sets: that is, if $\varphi \in C_c^\infty(M)$ is as before, i.e. $\varphi \equiv 1$ near $\partial M$ and supported near $\partial M$, then $\varphi K \in \Psi_0^{-\infty,\mathcal{F}}(M)$, with $\mathcal{F} = (F_{10}, F_{01}, F_{11})$ as in (3.5.2). Recall that $F_{10} = \bigcup_{j \geq 0} \overline{\{(n+j, p_j)\}}$ with $p_0 = 0$ and denote $F_{10}^\ell = \bigcup_{j \geq \ell} \overline{\{(n+j, p_j)\}}$ for $\ell \in \mathbb{N}_0$. Now let $P_0 = (x\partial_x - n) \in \mathrm{Diff}_0^1(M)$; $P_0$ lifts to $M_0^2$ to a $C^\infty$ vector field that takes the form $(s\partial_s - n)$ near $B_{10}$, where as before $s$ is a defining function for $B_{10}$. Then $K_0 := P_0(\varphi K) \in \Psi_0^{-\infty, F_{10}^1, F_{01}, F_{11}}(M)$, that is, the term of order $n$ in the expansion of $\varphi K$ at the left face of $M_0^2$ is removed. This allows us to show that $K_0 u \in x^\tau H_b^\infty(M; \frac{|dxdy|}{x})$ if $\tau < n+1$: by an inductive argument using commutators as in Lemma 3.5.2, one sees that if $P \in \mathrm{Diff}_b^m(M)$, $m \geq 0$, then $PK_0 u = \sum_{j=1}^{J_m} Q_j P_j u$, where $P_j \in \mathrm{Diff}_b^m(M)$ and $Q_j \in \Psi_0^{-\infty, F_{10}^1, F_{01}, F_{11}}(M)$. Recall that $P'u \in x^{\tau'} H_b^\infty(M; \frac{|dxdy|}{x}) \subset x^{\tau'} L^2(M; \frac{|dxdy|}{x})$ for $P' \in \mathrm{Diff}_b^m(M)$, $m \geq 0$ and $\tau' < n$ as observed earlier. Thus Proposition

3.2.4 together with the fact that $F_{10}^1 \geq n+1-\varepsilon$ for all $\varepsilon > 0$ and $F_{11} \geq 1$ implies that for $P \in \mathrm{Diff}_b^m(M)$, $PK_0u \in x^\tau L^2(M; \frac{|dxdy|}{x})$ if $\tau < n+1$ and $m \geq 0$, thus $K_0 u \in x^\tau H_b^\infty(M; \frac{|dxdy|}{x})$ for such $\tau$. This implies that $(K_0u)_{\mathcal{M}}(\zeta)$ is holomorphic on the half plane $\{\mathrm{Re}(\zeta) > -n-1\}$ with values in functions smooth in $y$. Since $(P_0(\varphi u))_{\mathcal{M}} = (-\zeta - n)(\varphi u)_{\mathcal{M}}$, $(\varphi u)_{\mathcal{M}} = (-\zeta-n)^{-1}(K_0u)_{\mathcal{M}}$ and we conclude that $(\varphi u)_{\mathcal{M}}$ extends meromorphically on the half plane $\{\mathrm{Re}(\zeta) > -n-1\}$, with a pole of order 1 at $\zeta = -n$ and values in smooth functions on $\partial M$. Computing the inverse Mellin transform on the line $\{\mathrm{Re}(\zeta) = -n-1+\varepsilon\}$, where $\varepsilon > 0$ is small (note that on such a line $(-\zeta-n)^{-1}(K_0u)_{\mathcal{M}}$ depends smoothly on $y$ and is in $L^2(\{\mathrm{Re}(\zeta) = -n-1+\varepsilon\}; |d\zeta|)$ for each $y$), we recover the leading term of the expansion of $u$: near $\partial M$

$$u(x,y) = a_{00}(y)x^n + v, \quad a_{00} \in C^\infty(\partial M),\ v \in x^\tau H_b^\infty\big(M; \frac{|dxdy|}{x}\big),\ \tau < n+1.$$

Now suppose that we have recovered the asymptotic expansion of $u$ up to the $m$-th exponent that appears in the index set $E$ and corresponds to powers of $x$, where $E$ is defined in the statement of the proposition. It will be convenient to write $E = \bigcup_{j\geq 0} \overline{\{(n+j, r_j)\}}$ and $F_{10} + mF_{11} = \bigcup_{j\geq m} \overline{\{(n+j, r_j^m)\}}$, so that $r_j + 1 = \sum_{m=0}^{j}(r_j^m + 1)$.[6] Note that $r_j^0 = p_j$ and also $r_0 = r_0^0 = p_0 = 0$. Thus suppose that we have

$$\begin{aligned} u =& u_m + v \text{ where } v \in x^\tau H_b^\infty\big(M; \frac{|dxdy|}{x}\big), \quad \tau < n+m, \\ &\text{and near } \partial M \quad u_m(x,y) = \sum_{j=0}^{m-1}\sum_{k=0}^{r_j} a_{jk}(y)x^{n+j}|\log x|^k, \quad a_{jk} \in C^\infty(\partial M). \end{aligned} \tag{3.5.5}$$

We will show that (3.5.5) holds for $m+1$; then by induction we will be done.

By (3.5.5), $(\varphi u)_{\mathcal{M}}$ is meromorphic on the half plane $\{\mathrm{Re}(\zeta) > -n-m\}$ with poles of order $r_j + 1$ at $\zeta = -n-j$ for $0 \leq j \leq m-1$. Set $P_j = (x\partial_x - n - j) \in \mathrm{Diff}_0^1(M)$ and write $K_m := \prod_{j=0}^m P_j^{p_j+1}(\varphi K)$; then $K_m \in \Psi_0^{-\infty, F_{10}^{m+1}, F_{01}, F_{11}}(M)$ as before. Now by (3.5.5)

$$\prod_{j=0}^{m} P_j^{p_j+1}(\varphi u) = K_m u_m + K_m v. \tag{3.5.6}$$

[6] Here the notation $r_j^m$ does not indicate raising to a power: the $r_j^m$ can be written explicitly in terms of $m$, the $p_j$, and the largest powers of logarithmic factors in an expansion induced by $F_{11}$, but we do not need such an explicit expression.

Since $K_m \in \Psi_0^{-\infty,F_{10}^{m+1},F_{01},F_{11}}(M)$ with $F_{11} \geq 1$ and $F_{10}^{m+1} \geq n+m+1-\varepsilon$ for all $\varepsilon > 0$, the fact that $v \in x^\tau H_b^\infty\big(M; \frac{|dxdy|}{x}\big)$, for $\tau < n+m$ implies that $K_m v \in x^\tau H_b^\infty(M; \frac{|dxdy|}{x})$ for $\tau < n+m+1$ using the same commutator argument as before and Proposition 3.2.4. Moreover, it follows by Proposition 3.2.2 that $K_m u_m \in \mathcal{A}_{phg}^G(M)$, where

$$G = F_{10}^{m+1}\overline{\cup}\Big(\overline{\bigcup}_{j=0}^{m-1}(F_{10}+jF_{11})+F_{11}\Big)\subset F_{10}^{m+1}\overline{\cup}\Big(\overline{\bigcup}_{k=1}^{m}(F_{10}+kF_{11})\Big) =: G', \qquad (3.5.7)$$

where the inclusion follows from Lemma 3.5.4. Thus $K_m u_m \in \mathcal{A}_{phg}^{G'}(M)$. Upon taking the Mellin transform in (3.5.6),

$$\prod_{j=0}^{m}(-\zeta-n-j)^{p_j+1}(\varphi u)_{\mathcal{M}}(\zeta) = (K_m u_m)_{\mathcal{M}}(\zeta) + (K_m v)_{\mathcal{M}}(\zeta), \qquad (3.5.8)$$

where $(K_m v)_{\mathcal{M}}(\zeta)$ is holomorphic in $\{\mathrm{Re}(\zeta) > -n-m-1\}$ (with values in $C^\infty(\partial M)$). On the other hand, for $1 \leq j \leq m$, $(K_m u_m)_{\mathcal{M}}(\zeta)$ has a pole of order $\sum_{k=1}^{j}(r_j^k+1)$ at $\zeta = -n-j$. Note that the index set $F_{10}^{m+1}$ in (3.5.7) does not contribute any poles in the open half plane $\{\mathrm{Re}(\zeta) > -n-m-1\}$. Thus upon dividing we find that $(\varphi u)_{\mathcal{M}}(\zeta)$ is meromorphic on the half plane $\{\mathrm{Re}(\zeta) > -n-m-1\}$ with values in $C^\infty(\partial M)$ and poles of order $p_j+1+\sum_{k=1}^{j}(r_j^k+1) = (r_j^0+1)+\sum_{k=1}^{j}(r_j^k+1) = r_j+1$ at $\zeta = -n-j$, $0 \leq j \leq m$. Taking the inverse Mellin transform of (3.5.8) on a vertical line $\{\mathrm{Re}(\zeta) = -n-m-1+\varepsilon\}$ for small $\varepsilon > 0$ similarly to the first inductive step we obtain (3.5.5) for $m+1$ and we are done. $\square$

**Remark 3.5.7.** *It follows from* (3.5.7) *that the index set $E$ in the statement of Proposition* (3.5.6) *allows for higher powers of logarithmic factors than it needs to, but its form suffices for our needs.*

We will need the following standard result from functional analysis (see [SU04] for a proof):

**Lemma 3.5.8.** *Let $X$, $Y$, $Z$ be Banach spaces, and let $A : X \to Y$ be bounded and injective. If there exists a compact operator $K : X \to Z$ such that*

$$\|u\|_X \leq C\left(\|Au\|_Y + \|Ku\|_Z\right), \quad u \in X$$

*for some constant $C$, then there exists a constant $C'$ such that*

$$\|u\|_X \leq C'\|Au\|_Y, \quad u \in X.$$

We now prove the main theorem:

*Proof of Theorem 3.* Let $u \in x^\delta L^2(M; dV_g) \cap \ker(\mathcal{N}_g)$, $\delta \in (-n/2, n/2)$. We claim that $u = 0$. Note that the X-ray transform is well defined on such a $u$: by Corollary 3.3.5, $Iu \in \langle\eta\rangle_h^{-\delta'} L^2(\partial_- S^*M; d\lambda_\partial)$, $\delta' < \min\{\delta, 0\}$. By Proposition 3.5.6, $u \in \mathcal{A}^E_{phg}(M)$, $E \geq n$. In particular, $u \in x^\delta L^2(M; dV_g)$ for $\delta < n/2$. Now by (3.1.6) and the discussion immediately after it we find

$$0 = (\mathcal{N}_g u, u)_{L^2(M;dV_g)} = (I^* Iu, u)_{L^2(M;dV_g)} = \|Iu\|^2_{L^2(\partial_- S^*M; d\lambda_\partial)}.$$

This implies that $Iu = 0$. Then one checks that the proof of Theorem 1 in [GGS$^+$], which shows injectivity of $I$ on $xC^\infty(M)$, also applies for polyhomogeneous functions in $\mathcal{A}^E_{phg}(M)$, $E \geq 1$. More specifically, by the proof of Proposition 3.15 there it follows that for $u \in \mathcal{A}^E_{phg}(M) \cap \ker I$ one has the stronger result $u \in \dot{C}^\infty(M)$ (i.e. $u$ vanishes to infinite order at the boundary). Then the injectivity argument using Pestov identities in the proof of Theorem 1 in the same paper yields $u \equiv 0$. We have shown that $\mathcal{N}_g$ is injective on $x^\delta L^2(M; dV_g)$, $\delta > -n/2$. Now by Proposition 3.5.1 we have

$$\|u\|_{x^\delta H^s_0(M;dV_g)} \leq C\left(\|\mathcal{N}_g u\|_{x^\delta H^{s+1}_0(M;dV_g)} + \|Ku\|_{x^\delta H^s_0(M;dV_g)}\right), \quad \delta \in (-n/2, n/2),\ s \geq 0,$$

where $K : x^\delta H^s_0(M; dV_g) \to x^\delta H^s_0(M; dV_g)$ is compact. Thus Lemma 3.5.8 implies

$$\|u\|_{x^\delta H^s_0(M;dV_g)} \leq C'\|\mathcal{N}_g u\|_{x^\delta H^{s+1}_0(M;dV_g)}, \quad \delta \in (-n/2, n/2),\ s \geq 0,$$

which is the claimed estimate. $\square$

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